Weak L^p

Def: (Distribution Function)

Let $f : X \to \mathbb{C}$ be measurable. We define the distribution function

$$\lambda_f : (0,\infty) \longrightarrow [0,\infty].$$

by

$$\lambda_f(\alpha) := \mu\left(\{x \in X \mid |f(x)| > \alpha\}\right).$$

Prop:

a.) λ_f is decreasing and right continuous.

b.) $|f| \le |g|$ implies $\lambda_f \le \lambda_g$

c.) $|f_n| \nearrow |f|$ implies $\lambda_{f_n} \nearrow \lambda_f$

d.) If $f = g + h$, then $\lambda_f(\alpha) \le \lambda_g(\tfrac{1}{2}\alpha) + \lambda_h(\tfrac{1}{2}\alpha)$.

Pf: Straight-forward.

Prop: If $\lambda_f(\alpha) < \infty$ for all $\alpha > 0$ and ϕ is a nonnegative Borel measurable function on $[0, \infty)$, then

$$\int_X \phi \circ |f| \, d\mu = -\int_0^\infty \phi(\alpha) \, d\lambda_f(\alpha).$$

Pf: Let ν_f be defined by

$$\nu_f([a,b]) := \lambda_f(b) - \lambda_f(a) = -\mu(\{a < |f| \leq b\})$$
$$= -\mu(|f|^{-1}((a,b])).$$

$\Rightarrow$ $\nu_f(E) = -\mu(|f|^{-1}(E))$ for any E Borel.

Then for χ_E

$$\int \chi_E \circ |f| \, d\mu = \int \chi_{|f|^{-1}(E)} \, d\mu$$
$$= \mu(|f|^{-1}(E))$$
$$= -\nu_f(E)$$
$$= -\int \chi_E \, d\nu_f = -\int \chi_E \, d\lambda_f$$

$\square$

Cor: $\int |f|^p \, dm = -\int_0^\infty \alpha^p \, d\lambda_f(\alpha)$

and $\int |f|^p \, dm = p \int_0^\infty \alpha^{p-1} \lambda_f(\alpha) \, d\alpha$

Def: For $1 \leq p < \infty$ define

$$[f]_p := \left(\sup_{\alpha > 0} \alpha^p \lambda_f(\alpha) \right)^{\frac{1}{p}}$$

$$\text{weak-}L^p := \left\{ f : X \to \mathbb{C} \;\middle|\; \begin{array}{l} f \text{ measurable} \\ [f]_p < \infty \end{array} \right\}.$$

Note: $[f]_p$ is not a norm.

Note: Chebyshev $\Rightarrow$ $[f]_p \leq \|f\|_p$.

Why do we care about weak-L^p spaces?

At times, unbounded operators on L^p spaces can be shown to be bounded in the following weak sense:

$$[Tf]_p \leq C \|f\|_p.$$

For example, If M = Hardy-Littlewood Maximal Function. Then

$$[Mf]_1 \leq \|f\|_1.$$

Moreover, this weak boundedness can be used to establish actual boundedness on other L^p spaces via Interpolation.

Interpolation

Observe: For $1 \le p < q < r \le \infty$

$$L^p \cap L^r \subset L^q \subset L^p + L^r$$

Def: (1) T is called sublinear if

- $|T(f+g)| \le |Tf| + |Tg|$
- $|T(cf)| = c\,|Tf|$. for $c>0$

(2) A sublinear map, T, is strong type (p,q) if $L^p(\omega) \subset D(T)$, $T : L^p(\omega) \to L^q(\omega)$ and $\exists C>0$ s.t.

$$\|Tf\|_q \le C\|f\|_p \qquad \text{for all } f \in L^p$$

(3) A sublinear map, T, is weak type (p,q) if $L^p(\omega) \subset D(T)$, $T : L^p(\omega) \to \text{weak-}L^q$ and $\exists C>0$ such that

$$[Tf]_q \le C\|f\|_p \qquad \forall f \in L^p.$$

(Note: weak type (p,∞) = strong type (p,∞).)

Observe: T strong type $(p,q) \Rightarrow T$ weak type (p,q).

The Marcinkiewicz Interpolation Thm

Thm: Let $(X,\mathcal{M},\mu)$ and $(Y,\mathcal{N},\nu)$ be measure spaces.

Let
$$p_0, p_1, q_0, q_1 \in [1,\infty].$$

$$p_0 \le q_0, \quad p_1 \le q_1 \quad \text{and} \quad q_0 \ne q_1$$

Define
$$\frac{1}{p} := \frac{1-t}{p_0} + \frac{t}{p_1}, \quad \frac{1}{q} := \frac{1-t}{q_0} + \frac{t}{q_1}$$

for $t \in (0,1)$.

If T is a sublinear map from $L^{p_0} + L^{p_1}$ to the space of measurable functions on Y that is both

weak type (p_0,q_0)

and weak type (p_1,q_1)

Then T is strong type (p,q).

We will not prove this.

Applications

Boundedness of Hardy-Littlewood Maximal Function.

<u>Recall:</u> For $f \in L^1_{loc}(\mathbb{R}^d)$

$$Mf(x) := \sup \left\{ \frac{1}{m(B)} \int_B |f| \, dm \;\middle|\; \begin{matrix} x \in B \\ B \text{ is a ball} \end{matrix} \right\}.$$

<u>Thm:</u>

$$[Mf]_1 \leq C \|f\|_1$$

<u>pf:</u> $5r$ covering lemma.

<u>Claim:</u>
- Mf is sublinear
- $\|Mf\|_\infty \leq C \|f\|_\infty$

<u>Corollary to Marcinkiewicz Interpolation</u>

The Hardy-Littlewood Maximal Function is strong-type (p,p) for $1 < p < \infty$.