The Riesz Representation Thm.

<u>Thm:</u> If L is a positive linear functional on $C_c(X)$, then there is a unique Radon measure μ on X such that

$$L(f) = \int f \, d\mu \quad \text{for all } f \in C_c(X)$$

Moreover, μ satisfies

$$\mu(U) = \sup\left\{ L(f) \;\middle|\; f \in C_c(X),\ \substack{f \le \chi_U \\ \operatorname{supp}(f) \subset U} \right\}$$

for all $U \subset X$ open

and

$$\mu(K) = \inf\left\{ L(f) \;\middle|\; f \in C_c(X),\ f \ge \chi_K \right\}$$

for all $K \subset X$ compact.

<u>Proof:</u>

Uniqueness: Urysohn Lemma plus regularity.

Existence: (This is difficult since $\chi_E \notin C_c(X)$)

For $U \subset X$ open define

$$\mu(U) = \sup\{L(f) \mid f \in C_c(X), f \le \chi_U, \mathrm{supp}(f) \subset U\}$$

and for any $E \in \mathcal{P}(X)$ define the outer measure

$$\mu^*(E) := \inf\{\mu(U) \mid E \subset U, U \text{ open}\}.$$

Claim 1: μ^* is an outer measure

First show that $\mu(U) \le \sum_{j=1}^{\infty} \mu(V_j)$

$U \subset \bigcup_{j=1}^{\infty} V_j$, V_j open.

Let $f \in C_c(X)$ $f \le \chi_U$, $\mathrm{supp}(f) \subset U$.

Let $K := \mathrm{supp}(f)$, then K is compact.

Therefore, $K \subset \bigcup_{j=1}^{n} V_j$ for some n.

There exists a partition of unity, $\{g_j\}$, on K subordinate to $\{V_j\}_{j=1}^{n}$

$\Rightarrow \quad g_j \le \chi_{V_j}$, $\operatorname{supp}(g_j) \subset V_j$ and

$$\sum_{j=1}^{n} g_j = 1 \quad \text{on } K.$$

Then $$f = f \cdot \sum g_j = \sum f \cdot g_j$$

and $f \cdot g_j \le \chi_{V_j}$, $\operatorname{supp}(f \cdot g_j) \subset V_j$.

$$\Rightarrow \quad L(f) = \sum_{j=1}^{n} L(f g_j) \le \sum_{j=1}^{n} \mu(V_j) \le \sum_{j=1}^{\infty} \mu(V_j)$$

$$\Rightarrow \quad \mu(U) \le \sum_{j=1}^{\infty} \mu(V_j)$$

Countable subadditivity for all sets, $E \subset \mathcal{P}(X)$, follows by standard argument

Claim 2: Open sets are μ^*-measurable

We want to show $\mu^*(E \cap U) + \mu^*(E \setminus U) \le \mu^*(E)$ for all $E \in \mathcal{P}(X)$.

First, suppose E is open. Then $E \cap U$ is open and thus

$$\mu^*(E \cap U) = \mu(E \cap U)$$

Therefore, there exists $f \in C_c(X)$ such that

$$f \le \chi_{E \cap U}, \quad \mathrm{supp}(f) \subset E \cap U \quad \text{and}$$

$$\mu^*(E \cap U) < L(f) + \varepsilon.$$

Moreover, $E \setminus U \subset E \setminus \mathrm{supp}(f)$ and

$E \setminus \mathrm{supp}(f)$ is open, so $\exists g \in C_c(X)$

s.t. $g \le \chi_{E \setminus \mathrm{supp}(f)}$ $\mathrm{supp}(g) \subset E \setminus \mathrm{supp}(f)$

and

$$\mu^*(E \setminus \mathrm{supp}(f)) < L(g) + \varepsilon$$

$\overset{\text{by claim 1}}{\Longrightarrow}$ $\mu^*(E \setminus U) < L(g) + \varepsilon.$

Which implies

$$\mu^*(E \cap U) + \mu^*(E \setminus U) < L(f) + L(g) + 2\varepsilon$$

$$= L(f+g) + 2\varepsilon \le \mu(E) + 2\varepsilon$$

$$= \mu^*(E) + 2\varepsilon.$$

$$\Rightarrow \mu^*(E \cap U) + \mu^*(E \setminus U) \le \mu^*(E).$$

Let E be arbitrary and $\mu^*(E) < \infty$.

Outer regularity implies.

$$\mu^*(E) + \varepsilon \geq \mu(V) \geq \mu^*(V \setminus U) + \mu^*(V \cap U)$$

$$\geq \mu^*(E \setminus U) + \mu^*(E \cap U)$$

This implies Borel sets are measurable and $\mu := \mu^*|_{\mathcal{B}_X}$ is a measure

Claim 3: $\mu(K) = \inf\{L(f) \mid f \in C_c(X),\ f \geq \chi_K\}$.

1. $\mu(K) \leq \inf\{L(f) \mid f \in C_c(X),\ f \geq \chi_K\}$.

Let $f \in C_c(X)$, $f \geq \chi_K$.

For $\varepsilon > 0$, consider

$$U_\varepsilon := f^{-1}((1-\varepsilon, \infty))$$

$\Rightarrow$ U_ε is open and $K \subset U_\varepsilon$

Let $g \in C_c(X)$ satisfy

$$g \leq \chi_{U_\varepsilon} \quad \text{and} \quad \operatorname{supp}(g) \subset U_\varepsilon.$$

Then $g \leq (1-\varepsilon)^{-1} f$ and

$L(g) \leq (1-\varepsilon)^{-1} L(f)$.

Therefore,

$$\mu(U_\varepsilon) \leq (1-\varepsilon)^{-1} L(f)$$

And, by monotonicity,

$$\mu(K) \leq (1-\varepsilon)^{-1} L(f) \quad \text{for all } \varepsilon > 0.$$

$$\Rightarrow \quad \mu(K) \leq L(f).$$

② $\underline{\inf \{L(f) \mid} \; f \in C_c(X),\, f \geq \chi_K\} \leq \mu(K)$

Let $\varepsilon > 0$ and U open such that $K \subset U$ and

$\mu(K) \geq \mu(U) - \varepsilon$. Urysohn's Lemma implies there exists $g \in C_c(X)$ such that

$$\chi_K \leq g \leq \chi_U, \quad \operatorname{supp}(g) \subset U$$

Therefore,

$$\mu(K) \geq \mu(U) - \varepsilon \geq L(g) - \varepsilon \geq \inf\{L(f)\} - \varepsilon.$$

Claim 4: $L(f) = \int f \, d\mu$ for $f \in C_c(X)$.

Suffices to show that

$$L(f) = \int f \, d\mu \quad \text{for} \quad f \in C_c(X)$$

$$f : X \to [0,1].$$

Let $N \in \mathbb{N}$, $j \in \{1, \dots, N\}$ and

$$K_j := \{x \in X \mid f(x) \geq j/N\} \qquad j = 1, \dots, N.$$

$$K_0 := \operatorname{supp}(f)$$

and

$$f_j(x) := \begin{cases} 0 & , \ x \notin K_{j-1} \\ f(x) - \frac{j-1}{N} & , \ x \in K_{j-1} \setminus K_j \\ \frac{1}{N} & , \ x \in K_j \end{cases}$$

Then $f = \sum_{j=1}^{N} f_j$ and $\frac{1}{N} \chi_{K_j} \leq f_j \leq \frac{1}{N} \chi_{K_{j-1}}$

$$\Rightarrow \quad \frac{1}{N} \mu(K_j) \leq \int f_j \, d\mu \leq \frac{1}{N} \mu(K_{j-1}).$$

For any open set $K_{j-1} \subset U$, $\operatorname{supp} f_j \subset U$

and $f_j \leq \frac{1}{N} \chi_U$

This implies

$$L(f_j) \leq \frac{1}{N}\mu(U)$$

Therefore, $$L(f_j) \leq \frac{1}{N}\mu(K_{j-1}).$$

By claim 3, $$\frac{1}{N}\mu(K_j) \leq L(f_j).$$

Thus,

$$\frac{1}{N}\sum_{j=1}^{N}\mu(K_j) \leq L(f) \leq \frac{1}{N}\sum_{j=1}^{N}\mu(K_{j-1})$$

Finally, $$\left|L(f) - \int f\,d\mu\right| \leq \frac{1}{N}\left(\sum_{j=1}^{N}\mu(K_{j-1}) - \sum_{j=1}^{N}\mu(K_j)\right).$$

$$= \frac{1}{N}\,\mu(K_0) - \mu(K_N)$$

$$\xrightarrow{N\to\infty} 0.$$

$\square$.

Let (X,τ) be compact, Hausdorff and define the space of measures

$$M(X) = \{\mu : \mathcal{B}_X \to \mathbb{C} \mid \mu \text{ Radon}\}.$$

$$\|\mu\| := |\mu|(X) < \infty$$

Riesz Rep. Thm implies $(C(X))^* = M(X)$.

Weak Convergence

We can now say that $\{f_n\}_{n=1}^{\infty} \subset L^p(\mu)$ (for $1 \leq p < \infty$) converges weakly to $f \in L^p(\mu)$ if

$$\int f_n g \, d\mu \xrightarrow{n\to\infty} \int f g \, d\mu \quad \text{for all } g \in L^q(\mu)$$

where $\frac{1}{q} + \frac{1}{p} = 1$.

$\{f_n\}_{n=1}^{\infty} \subset C_c(X)$ converges weakly to $f \in C_c(X)$ if

$$\int f_n \, d\mu \xrightarrow{n\to\infty} \int f \, d\mu \quad \text{for all}$$

Radon measures μ.

Finally, $\{\mu_n\}_{n=1}^{\infty} \subset M(X)$ converges weak-* to $\mu \in M$ if

$$\int g \, d\mu_n \to \int g \, d\mu \quad \text{for all } g \in C_c(X)$$

(Sometimes, it is said that μ_n converges weakly to μ)