# The Dual of $C_c(X)$

{Note: Recall that $C_c(X)$ is not complete unless $X$ is compact}

Let $X$ be a locally compact, Hausdorff topological space.

**Recall:**

Def: (Radon Measure)

A Borel measure, $\mu$, is Radon if

a.) $\mu(E) = \inf\{\mu(U) \mid E \subset U,\ U \text{ open}\}\ \forall E \in \mathcal{B}_X$.

b.) $\mu(U) = \sup\{\mu(K) \mid K \subset U,\ K \text{ compact}\}\ \forall U$ open

c.) $\mu(K) < \infty$ for all $K$ compact.

Def: (Positive Linear Functional)

A linear functional, $L$, on $C_c(X)$ is called <u>positive</u> if $L(f) \geq 0$ whenever $f \geq 0$

**Lemma:** If $L$ is a positive linear functional on $C_c(X)$, then for each compact $K \subset X$ $\exists$ constant $C_K$ such that

$$|L(f)| \leq C_K \|f\|_\infty$$

for all $f \in C_c(X)$ such that $\operatorname{supp}(f) \subset K$

**proof:** Suppose $f$ is real-valued.

Given $K$ compact, choose $\varphi \in C_c(X)$ satisfying $\varphi(x) \in [0,1]$ and

$$\varphi(x) = 1 \quad \text{for all } x \in K.$$

If $\operatorname{supp}(f) \subset K$, then $|f(x)| \leq \|f\|_\infty \varphi(x)$

$\Rightarrow$ $f(x) \leq \|f\|_\infty \varphi(x)$ and $-f(x) \leq \|f\|_\infty \varphi(x)$.

Therefore,

$$L(\|f\|_\infty \varphi - f) \geq 0 \quad \text{and} \quad L(\|f\|_\infty \varphi + f) \geq 0$$

which implies

$$L(f) \leq \|f\|_\infty L(\varphi) \quad \text{and} \quad -L(f) \leq \|f\|_\infty L(\varphi)$$

Moreover, $|L(f)| \leq \|f\|_\infty L(\varphi)$

Therefore, $C_K = L(\varphi) \geq 0$ $\square$.

# Partitions of Unity

**Def:** (Partition of Unity)

If $(X, \tau)$ is a top. space and $E \subset X$. A <u>partition of unity</u> on $E$ is a collection, $\{h_\alpha\}_{\alpha \in I}$, of continuous functions, $h_\alpha : X \to [0,1]$ such that

- Each $x \in X$ has a nbhd on which finitely many $h_\alpha$'s are nonzero.
- $\sum_{\alpha \in I} h_\alpha = 1$ for $x \in E$.

A partition of unity, $\{h_\alpha\}_{\alpha \in I}$, is <u>subordinate</u> to an open cover, $\{U_\beta\}_{\beta \in B}$, of $E$ if for each $\alpha \in I$ $\exists U_\beta$ with

$$\operatorname{supp}(h_\alpha) \subset U_\beta$$

**Prop:** (Existence of a Partition of Unity)
Let $X$ be a locally compact, Hausdorff space, $K \subset X$ compact and $\{V_j\}_{j=1}^n$ be an open cover of $K$. There is a partition of unity on $K$ subordinate to $\{V_j\}_{j=1}^n$ consisting of compactly supported functions.

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**pf:** For each $x \in K$, $x \in V_j$ for some $j$.
Local compactness $\Rightarrow$ $\exists N_x$ compact s.t. $x \in N_x \subset V_j$ and $\{\text{int}(N_x)\}_{x \in K}$ is an open cover of $K$. Therefore,

$$K \subset \bigcup_{m=1}^{M} N_{x_m} \quad \text{for some } \{x_m\}_{m=1}^{M} \subset K$$

Define $F_j := \bigcup_{N_{x_m} \subset V_j} N_{x_m}$, $F_j$ is compact and $F_j \subset V_j$

Urysohn's Lemma implies that there exists $g_1, \ldots, g_n \in C_c(X)$ such that

$$\chi_{F_j} \leq g \leq \chi_{V_j} \quad \text{and} \quad \text{supp}(g_j) \subset V_j$$

$$\Rightarrow \sum_{j=1}^{n} g_j \geq 1 \quad \text{on } K \quad \text{since } g_j \geq 1 \text{ on } F_j$$

Urysohn's Lemma again implies there exists $f \in C_c(X)$ such that

$f = 1$ on $K$ and

$$\operatorname{supp}(f) \subset \left\{ \sum_{j=1}^{n} g_j > 0 \right\}.$$

Let $g_{n+1} = 1 - f$

Then $\sum_{j=1}^{n+1} g_j > 0$ on $X$

Finally, define

$$h_j := \frac{g_j}{\sum_{k=1}^{n+1} g_k} \quad \text{for } j = 1, \dots, n$$

Therefore,

$$\operatorname{supp}(h_j) = \operatorname{supp}(g_j) \subset V_j$$

and $\sum_{j=1}^{n} h_j = 1$ on $K$

$\square$.

# The Riesz Representation Thm.

Thm: If $L$ is a positive linear functional on $C_c(X)$, then there is a unique Radon measure $\mu$ on $X$ such that

$$L(f) = \int f \, d\mu \quad \text{for all } f \in C_c(X)$$

Moreover, $\mu$ satisfies

$$\mu(U) = \sup\{ L(f) \mid f \in C_c(X),\ \underset{\operatorname{supp}(f) \subset U}{f \leq \chi_U} \}$$

for all $U \subset X$ open

and

$$\mu(K) = \inf\{ L(f) \mid f \in C_c(X),\ f \geq \chi_K \}$$

for all $K \subset X$ compact.

Proof:

Uniqueness: Urysohn Lemma plus regularity.

Existence: (This is difficult since $\chi_E \notin C_c(X)$)

For $U \subset X$ open define

$$\mu(U) = \sup\{L(f) \mid f \in C_c(X),\ f \leq \chi_U,\ \mathrm{supp}(f) \subset U\}$$

and for any $E \in \mathcal{P}(X)$ define the outer measure

$$\mu^*(E) := \inf\{\mu(U) \mid E \subset U,\ U \text{ open}\}.$$

Claim 1: $\mu^*$ is an outer measure

First show that $\mu(U) \leq \sum_{j=1}^{\infty} \mu(V_j)$

$U \subset \bigcup_{j=1}^{\infty} V_j$, $V_j$ open.

Let $f \in C_c(X)$ $f \leq \chi_U$, $\mathrm{supp}(f) \subset U$.

Let $K := \mathrm{supp}(f)$, then $K$ is compact.

Therefore, $K \subset \bigcup_{j=1}^{n} V_j$ for some $n$.

There exists a partition of unity, $\{g_j\}$, on $K$ subordinate to $\{V_j\}_{j=1}^{n}$

$\Rightarrow \quad g_j \le \chi_{V_j}$ , $\operatorname{supp}(g_j) \subset V_j$ and

$$\sum_{j=1}^{n} g_j = 1 \quad \text{on } K.$$

Then $f = f \cdot \sum g_i = \sum f \cdot g_j$

and $f \cdot g_j \le \chi_{V_j}$ , $\operatorname{supp}(f \cdot g_j) \subset V_j$.

$$\Rightarrow \quad L(f) = \sum_{j=1}^{n} L(f g_j) \le \sum_{j=1}^{n} \mu(V_j) \le \sum_{j=1}^{\infty} \mu(V_j)$$

$$\Rightarrow \quad \mu(U) \le \sum_{j=1}^{\infty} \mu(V_j)$$

Countable subadditivity for all sets, $E \subset \mathcal{P}(X)$, follows by standard argument

**Claim 2:** Open sets are $\mu^*$-measurable

We want to show $\mu^*(E \cap U) + \mu^*(E \setminus U) \le \mu^*(E)$ for all $E \in \mathcal{P}(X)$.

First, suppose $E$ is open. Then $E \cap U$ is open and thus

$$\mu^*(E \cap U) = \mu(E \cap U)$$