The Riesz Representation Thm

We will characterize the dual of $L^p(\mu)$ for $1 \le p < \infty$ and $C_c(X)$.

The Dual of $L^p(\mu)$ for $1 \le p < \infty$

Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space. Let $1 \le p < \infty$, $1 < q \le \infty$ $\frac{1}{p} + \frac{1}{q} = 1$. For $g \in L^q(\mu)$, define the linear functional

$$l_g(f) := \int f \bar{g} \, d\mu$$

Hölder's Inequality $\Rightarrow$ $|l_g(f)| \le \|g\|_q \|f\|_p$

$\Rightarrow$ $l_g \in (L^p(\mu))^*$ and $\|l_g\| \le \|g\|_q$.

Therefore there exists a continuous injection of $L^q(\mu)$ into $(L^p(\mu))^*$

Prop: $\|l_g\| = \sup_{\|f\|_p=1} |l_g(f)| = \|g\|_q$

for $1 \le q < \infty$

pf: Suffices to show $\|l_g\| \ge \|g\|_q$

By construction:

Let $f(x) := \dfrac{|g(x)|^{q-1} g(x)}{\|g\|_q^{q-1}}$

Then $l_g(f) = \int \dfrac{|g|^q}{\|g\|_q^{q-1}} = \|g\|_q.$

and $\|f\|_p = \int \dfrac{|g|^{(q-1)p}}{\|g\|_q^{(q-1)p}} d\mu = \dfrac{\int |g|^q d\mu}{\|g\|_q^q} = 1$ $\square$.

Prop: Suppose $g \in L^1_{loc}(\mu)$

If $M_q(g) := \sup\{ |\int f g| : f \text{ simple}, \|f\|_p = 1\} < \infty$

Then $g \in L^q(\mu)$ and $\|g\|_q = M_q(g)$.

pf: Let $\{\phi_n\}_{n=1}^{\infty}$ be a sequence of simple functions such that $\phi_n \to g$ pointwise and $|\phi_n| \le |g|$ $\arg(\phi) = \arg(g)$.
Since (X, M, μ) is σ-finite, $\exists \{E_n\}_{n=1}^{\infty}$ such that $\mu(E_n) < \infty$ for all n. and

$$X = \bigcup_{n=1}^{\infty} E_n.$$

Define $g_n = \phi_n \chi_{E_n}$. Then $g_n \to g$ pointwise μ-a.e. and $\|g_n\|_q < \infty$.

By previous proposition, $\exists f_n$ such that $\|f_n\|_p = 1$

By construction, f_n is simple as well

Then, by Fatou's Lemma

$$\|g\|_q \le \liminf_{n\to\infty} \|g_n\|_q = \liminf_{n\to\infty} \int |f_n g_n|$$

$$\le \liminf_{n\to\infty} \int |f_n g| \le M_q(g)$$

Obviously, $M_q(g) \le \|g\|_q$ by Hölder's ineq

$\square$.

Thm: (Riesz Rep. Thm)

Let $1<p<\infty$, $\frac{1}{p}+\frac{1}{q}=1$, let $(X,\mathcal{M},\mu)$ be a σ-finite measure space. For each $L \in (L^p(\mu))^*$, $\exists g \in L^q(\mu)$ such that

$$L(f) = \int f \bar{g}\, d\mu \quad \text{for all } f \in L^p(\mu)$$

and $\|L\| = \|g\|_q$

Therefore, $L^q(\mu)$ is isometrically isomorphic to $(L^p(\mu))^*$.

Pf:

Idea: Use Radon-Nikodym-Lebesgue derivative to get $g \in L^1_{loc}$ then use previous proposition to show $g \in L^q$.

Finite Measure Case:

Let $\mu(X) < \infty$ and $L \in (L^p(\mu))^*$

Define $\nu: \mathcal{M} \to \mathbb{C}$ by

$$\nu(E) := L(\chi_E).$$

Claim: ν is a complex measure on $(X, \mathcal{M})$ since $\chi_E \in L^p(\mu)$.

Moreover,

$$|\nu(E)| = |L(\chi_E)| \le \|L\| \, \|\chi_E\|_{L^p} = \|L\| \, (\mu(E))^{1/p}$$

$\Rightarrow$ $\nu \ll \mu$.

Radon-Nikodym $\Rightarrow$ $\exists \, g \in L^1_{loc}(\mu)$ such that

$$\nu(E) = \int_E g \, d\mu := L(\chi_E).$$

Therefore,

$$L(f) = \int f g \, d\mu$$ for all simple functions f.

and $$\left|\int f g \, d\mu\right| = |L(f)| \le \|L\| \, \|f\|_p .$$

The previous proposition implies

$g \in L^q$ and

$$\|g\|_q = \sup_{\substack{\|f\|_p = 1 \\ \text{simple}}} \left| \int f g \, d\mu \right| = \|L\| \quad \square.$$

The Dual of $C_c(X)$

{Note: Recall that $C_c(X)$ is not complete unless X is compact}

Let X be a locally compact, Hausdorff topological space.

Recall:

Def: (Radon Measure)

A Borel measure, μ, is Radon if

a.) $\mu(E) = \inf\{\mu(U) \mid E \subset U, U \text{ open}\}$ $\forall E \in \mathcal{B}_X$.

b.) $\mu(U) = \sup\{\mu(K) \mid K \subset U, K \text{ compact}\}$ $\forall U$ open

c.) $\mu(K) < \infty$ for all K compact.

Def: (Positive Linear Functional)

A linear functional, L, on $C_c(X)$ is called <u>positive</u> if $L(f) \geq 0$ whenever $f \geq 0$

Lemma: If L is a positive linear functional on $C_c(X)$, then for each compact $K \subset X$ $\exists$ constant C_K such that

$$|L(f)| \leq C_K \|f\|_\infty$$

for all $f \in C_c(X)$ such that $\mathrm{supp}(f) \subset K$

proof: Suppose f is real-valued.
Given K compact, choose $\varphi \in C_c(X)$ satisfying $\varphi(x) \in [0,1]$ and

$$\varphi(x) = 1 \quad \text{for all } x \in K.$$

If $\mathrm{supp}(f) \subset K$, then $|f(x)| \leq \|f\|_\infty \varphi(x)$

$\Rightarrow$ $f(x) \leq \|f\|_\infty \varphi(x)$ and $-f(x) \leq \|f\|_\infty \varphi(x)$.

Therefore,

$$L(\|f\|_\infty \varphi - f) \geq 0 \quad \text{and} \quad L(\|f\|_\infty \varphi + f) \geq 0$$

which implies

$$L(f) \leq \|f\|_\infty L(\varphi) \quad \text{and} \quad -L(f) \leq \|f\|_\infty L(\varphi)$$

Moreover, $|L(f)| \leq \|f\|_\infty L(\varphi)$

Therefore $C_K = L(\varphi) \geq 0$ $\square$.