Important Examples

① Sequence Spaces.

$\ell^p(\mathbb{N}) := L^p(\mu)$ where μ = counting measure

$X = \mathbb{N}$ and $\mathcal{M} = \mathcal{P}(\mathbb{N})$.

Observe: $\ell^p(\mathbb{N}) \subset \ell^q(\mathbb{N})$ for $p \leq q$.

$$\begin{aligned}
\sum |a_n|^q &= \sum |a_n|^p |a_n|^{q-p} \\
&\leq \left(\sum |a_n|^p\right) \sup_n |a_n|^{q-p} \\
&= \left(\sum |a_n|^p\right) \left(\sup_n |a_n|^p\right)^{\frac{q-p}{p}} \\
&\leq \left(\sum |a_n|^p\right) \left(\sum |a_n|^p\right)^{\frac{q-p}{p}} \\
&= \left(\sum |a_n|^p\right)^{q/p}
\end{aligned}$$

② Finite Measure Spaces

Let $\mu(X) < \infty$

Then $L^p(\mu) \subset L^q(\mu)$ for $p \geq q$.

By Jensen's Inequality

$$\begin{aligned}
\|f\|_{L^q(\mu)} &= \left(\int |f|^q \, d\mu\right)^{1/q} = \mu(X)^{1/q} \left(\frac{1}{\mu(X)} \int |f|^q \, d\mu\right)^{1/q} \\
&\leq \mu(X)^{1/q} \left(\frac{1}{\mu(X)} \int |f|^p \, d\mu\right)^{1/p} \\
&= \mu(X)^{1/q - 1/p} \|f\|_{L^p(\mu)}.
\end{aligned}$$

Mollifiers / Approximate Identities

Def: The family of bounded continuous functions, $\{\Phi_N\}_{N=1}^{\infty}$ defined on $\mathbb{R}^d$ is called an <u>approximate identity</u> if

a.) For all N,
$$\int \Phi_N \, dm = 1.$$

b.) $\sup_N \|\Phi_N\|_1 < \infty$

c.) $\forall \delta > 0$, $\lim_{N\to\infty} \int_{\|x\|>\delta} |\Phi_N(x)| \, m(dx) = 0.$

Examples

① $\Phi_N(x) = c_d N (1 - \|Nx\|)_+$

② $\Phi_N(x) = C_{N,d} \exp\left(\frac{1}{\|Nx\|^2 - 1}\right).$

Then $\{\Phi_N\} \subset C^\infty([-1,1]).$

Idea: $\Phi_N \xrightarrow{N\to\infty} \delta_0$ in the weak-* sense.

Thm: Let $g \in C_c(\mathbb{R}^d)$ and $\{\Phi_N\}$ be an approximate identity

Then $\Phi_N * g \xrightarrow{N\to\infty} g$ uniformly.

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pf: $g \in C_c(\mathbb{R}^d) \Rightarrow g$ is uniformly continuous. Let $C := \sup_N \|\Phi_N\|_1 < \infty$

Let $\varepsilon > 0$, choose $\delta > 0$ such that

$\|x-y\| < \delta \implies |g(x) - g(y)| < \varepsilon/100C$

Choose M large enough so that

$N > M$ implies $\displaystyle\int_{\|x\| > \delta} |\Phi_N(x)|\, dm < \varepsilon/100C$

Then

$$\|\Phi_N * g - g\|_\infty = \sup_{x \in \mathbb{R}^d} \left| \int_{\mathbb{R}^d} \Phi_N(t)\, g(x-t)\,dt - g(x) \right|$$

$$= \sup_x \left| \int \Phi_N(t) g(x-t)\,dt - \int \Phi_N(t) g(x)\,dt \right|$$

$$= \sup_x \left| \int \Phi_N(t) \big(g(x-t) - g(x)\big)\,dt \right|$$

$$\leq \sup_x \int_{\|t\|<\delta} |\Phi_N(t)|\,|g(x-t)-g(x)|\,dt$$

$$+ \sup_x \int_{\|t\|\geq\delta} |\Phi_N(t)|\,|g(x-t)-g(x)|\,dt$$

$$\leq \frac{\varepsilon}{100c}\|\Phi_N\|_1 + 2\|g\|_\infty \frac{\varepsilon}{100c} \to 0$$

$\square$.

Observation: $g \in C^k \Rightarrow f * g \in C^k$

Thm: $C^\infty(\mathbb{R}^d)$ is dense in $L^p(\mathbb{R}^d)$ for $1 \leq p < \infty$.

Pf: Let $\{\Phi_N\}_{N\geq 1}^\infty \subset C^\infty(\mathbb{R}^d)$ be an approximate identity and $\varepsilon > 0$.

Let $f \in L^p(\mathbb{R}^d)$, $\exists\, g \in C_c(\mathbb{R}^d)$ such that $\|f-g\|_{L^p} < \varepsilon/3$.

$\exists\, N$ s.t. $\|\Phi_N * g - g\|_\infty < \dfrac{\varepsilon}{3\mu(\operatorname{supp} g)^{1/p}}$

which implies

$$\|\Phi_N * g - g\|_{L^p} < \varepsilon/3.$$

Now

$$\begin{aligned}
\|\Phi_N * f - f\|_{L^p} &\leq \|\Phi_N * f - \Phi_N * g\|_{L^p} \\
&\quad + \|\Phi_N * g - g\|_{L^p} \\
&\quad + \|g - f\|_{L^p} \\
&\leq \left(\sup_N \|\Phi_N\|_1\right) \|f - g\|_{L^p} \\
&\quad + \varepsilon/3 + \varepsilon/3. \\
&< \left(\sup_N \|\Phi_N\|_1\right) \varepsilon/3 + \frac{2\varepsilon}{3} \to 0.
\end{aligned}$$

$\square$.