4 Important Inequalities

① Jensen's Inequality.

Thm: Let $(X, \mathcal{M}, \mu)$ be a finite measure space. Suppose $g \in L^1(\mu)$ and

$$g : X \to (a,b) \subset \mathbb{R}.$$

Suppose $F : (a,b) \to \mathbb{R}$ is a convex function, then

$$F\left(\frac{1}{\mu(X)} \int_X g \, d\mu\right) \leq \frac{1}{\mu(X)} \int_X F \circ g \, d\mu.$$

Proof:

Claim: If $t_0 \in (a,b)$, $\exists \beta \in \mathbb{R}$ such that

$$F(t) - F(t_0) \geq \beta (t - t_0) \quad \text{for all } t \in (a,b).$$

Observe: $\frac{1}{\mu(X)} \int_X g \, d\mu \in (a,b)$.

Let $$t_0 := \frac{1}{\mu(X)} \int_X g \, d\mu$$

$\Rightarrow \exists \beta$ such that

$$F(t) - F\left(\frac{1}{\mu(X)}\int_X g \, d\mu\right) \geq \beta\left(t - \frac{1}{\mu(X)}\int_X g \, d\mu\right).$$

for all $t \in (a,b)$

In particular,

$$\frac{1}{\mu(X)}\int_X F\circ g \, d\mu - F\left(\frac{1}{\mu(X)}\int_X g \, d\mu\right)$$

$$\geq \beta\left[\frac{1}{\mu(X)}\int_X g \, d\mu - \frac{1}{\mu(X)}\int_X g \, d\mu\right].$$

$$\Rightarrow \frac{1}{\mu(X)}\int_X F\circ g \, d\mu \geq F\left(\frac{1}{\mu(X)}\int_X g \, d\mu\right). \quad \square$$

② Hölder's Inequality.

<u>Thm:</u> Let $1 < p < \infty$, $1 < q < \infty$ and

$$\frac{1}{p} + \frac{1}{q} = 1.$$

If f and g are measurable functions on X, then

$$\|fg\|_1 \leq \|f\|_p \|g\|_q$$

(Equality holds iff $\alpha|f|^p = \beta|g|^q$ for some $\alpha, \beta \neq 0$).

Pf: WLOG let $\|f\|_p = \|g\|_q = 1$.

Lemma: (GM $\leq$ AM)

If $a, b \geq 0$ and $\lambda \in (0,1)$, then

$$a^\lambda b^{1-\lambda} \leq \lambda a + (1-\lambda) b.$$

w/ equality iff $a = b$

Let $a = |f|^p$ and $b = |g|^q$

$$\Rightarrow \quad |fg| = a^{1/p} b^{1/q} \leq \frac{1}{p} a + \frac{1}{q} b = \frac{1}{p}|f|^p + \frac{1}{q}|g|^q.$$

$$\Rightarrow \quad \int |fg| \, d\mu \leq \frac{1}{p}\|f\|_p^p + \frac{1}{q}\|g\|_q^q = 1. \quad \square.$$

③ Minkowski Inequality

Thm: If $1 \le p < \infty$ and $f, g \in L^p(\mu)$ then

$$\|f+g\|_p \le \|f\|_p + \|g\|_p$$

pf: Let $p > 1$

Observe: $|f+g|^p \le (|f| + |g|)\,|f+g|^{p-1}$

Hölder implies

$$\int |f+g|^p \le \|f\|_p \, \| \, |f+g|^{p-1} \|_q + \|g\|_p \, \| \, |f+g|^{p-1} \|_q$$

Note: $\int |f+g|^{(p-1)q} = \int |f+g|^p$

and $1/q = 1 - \frac{1}{p}$

$$\Rightarrow \int |f+g|^p \le \left(\|f\|_p + \|g\|_p\right) \|f+g\|_p^{p-1}$$

$$\Rightarrow \left(\int |f+g|^p\right)^{1/p} \le \|f\|_p + \|g\|_p$$

$\square$.

④ Young's Inequality.

Consider $f \in L^1(\mathbb{R}^d)$, $g \in L^1(\mathbb{R}^d)$

Recall the definition of convolution of f and g:

$$(f * g)(x) = \int f(x-y)g(y)\,dm(y) = \int f(y)g(x-y)\,dm(y)$$

Claim: $\|f*g\|_{L^1(\mathbb{R}^d)} \leq \|f\|_{L^1(\mathbb{R}^d)} \|g\|_{L^1(\mathbb{R}^d)}$

This is the first case of Young's Inequality

Thm: Let $1 \leq p, q, r < \infty$, $1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}$.

If $f \in L^p(\mathbb{R}^d)$, $g \in L^q(\mathbb{R}^d)$, then

$$\|f*g\|_{L^r(\mathbb{R}^d)} \leq \|f\|_{L^p} \|g\|_{L^q}.$$

Pf: Case 1: $p = r = q = 1$.

Follows from claim.

Case 2: $q=1$, $p=r$.

Want to show that $\|f*g\|_{L^p} \le \|f\|_p \|g\|_{L^1}$

First observe that

$f \in L^p$, $g \in L^1 \Rightarrow |f|\,|g(x-\cdot)|^{1/p} \in L^p(\mathbb{R}^d)$
for all $x \in \mathbb{R}^d$.

Let p' satisfy $\frac{1}{p} + \frac{1}{p'} = 1$, then Hölder's inequality implies

$$\int |f(y)|\,|g(x-y)|\,dy = \int |f(y)|\,|g(x-y)|^{1/p}\,|g(x-y)|^{1/p'}\,dy$$

$$\le \left(\int |f(y)|^p\,|g(x-y)|\,dy\right)^{1/p} \left(\int |g|\,dy\right)^{1/p'}$$

$$\Rightarrow |f*g(x)| \le \|g\|_{L^1}^{1/p'} \left(|f|^p * |g|\right)^{1/p}(x).$$

Therefore,

$$\int |f*g(x)|^p\,dx \le \|g\|_{L^1}^{p/p'} \int \left(|f|^p * |g|\right)(x)\,dx$$

By case 1

$$\int |f*g(x)|^p\,dx \le \|g\|_{L^1}^{p/p'}\,\|f\|_p^p\,\|g\|_{L^1}.$$

$\Rightarrow \quad \|f * g\|_{L^p} \leq \|f\|_p \|g\|_1$

General Case: Exercise

$\square$.

Corollary: $L^p(\mu)$ is a Banach Space

pf:

1. $L^p(\mu)$ is a normed linear space

Follows from Minkowski inequality

2. $L^p(\mu)$ is complete

$L^p(\mu)$ is complete $\Leftrightarrow$ absolutely convergent series converge

Case 1: $p = \infty$, Absolutely convergent series

$\Longleftrightarrow$

μ-a.e. Cauchy sequence.

Case 2: $p \geq 1$

Suppose $\{f_n\}_{n=1}^{\infty} \subset L^p$ and

$$B = \sum_{n=1}^{\infty} \|f_n\|_p < \infty.$$

Let $G_N = \sum_{n=1}^{N} |f_n|$ and

$$\|G_N\|_p \leq \sum_{n=1}^{N} \|f_n\|_{L^p} \leq B \quad \text{for all } N.$$

By Monotone Convergence Theorem implies

$$\int \lim_{n\to\infty} \left(\sum_{n=1}^{N} |f_n| \right)^p = \lim_{N\to\infty} \int |G_N|^p \leq B^p < \infty.$$

$$\Rightarrow \quad G := \sum_{n=1}^{\infty} |f_n| \in L^p$$

$$\Rightarrow \quad \sum_{n=1}^{\infty} |f_n| \quad \text{converges } \mu\text{-a.e.}$$

$$\Rightarrow \quad F := \sum_{n=1}^{\infty} f_n \quad \text{converges } \mu\text{-a.e.}$$

By Dominated Convergence Theorem,

$$\lim_{N\to\infty} \int \left| F - \sum_{n=1}^{N} f_n \right|^p d\mu = 0 \quad \square.$$