Lecture 1

Functional Analysis

Motivation

1. In calculus, we are concerned with solving problems of the form

$$\begin{cases} \frac{d}{dt} f(t) = g(t) \\ f(0) = a \end{cases} \qquad f : \mathbb{R} \to \mathbb{R}$$

2. In ODE, we solve problems ~~of the form~~

$$\begin{cases} \frac{d}{dt} f(t) = G(f(t)) \\ f(0) = \underline{a} \in \mathbb{R}^d \end{cases} \qquad f : [0,\infty) \to \mathbb{R}^d$$

where $G : \mathbb{R}^d \to \mathbb{R}^d$ is "nice"

which leads to

$$\begin{cases} \frac{d}{dt} f(t) = [DG(\underline{0})] f(t) \\ f(0) = \underline{a} \in \mathbb{R}^d \end{cases}$$

If $DG(\underline{0})$ is diagonalizable, then via c.o.v.

$$\begin{cases} d/dt\, h(t) = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_d \end{bmatrix} h(t). \\ h(0) = \underline{b} \in \mathbb{R}^d \end{cases}$$

$$\frac{d}{dt} h_i(t) = \lambda_i h_i(t).$$

Now consider PDEs

$$\begin{cases} \frac{d}{dt} f(t,x) = G(f(t,x)) \\ f(0,x) = g(x) \in \text{some function space} \end{cases}$$

where G is a differential operator.

$\downarrow$ linearize

$$\begin{cases} d/dt\, f(t,x) = T(f(t,x)) \\ f(0,x) = g(x) \end{cases}$$

where T is a "linear operator"

Goals:

- Develop geometric/topological theory of spaces of functions
- Develop analytic theory of linear operators on function spaces.

Linear Spaces

Def: (Linear Spaces / Vector Spaces)

A set, X, is a <u>linear space</u> (vector space) over a Field, K, if

i.) $\exists\, 0 \in X$, $0 + x = x \quad \forall x \in X$

ii.) If $\alpha, \beta \in K$, $x, y \in X$ then $\alpha x + \beta y \in X$.

Def: (Normed Linear Space)

A <u>normed linear space</u>, $(X, \|\cdot\|)$, is a linear space, X, over $\mathbb{R}$ (or $\mathbb{C}$) with a function $\|\cdot\| : X \to [0, \infty)$ satisfying

i.) $\|x\| = 0 \iff x = 0$

ii.) $\|\alpha x\| = |\alpha| \cdot \|x\| \quad \forall x \in X, \alpha \in \mathbb{R}$

iii.) (Δ-ineq) $\|x + y\| \leq \|x\| + \|y\|$.

Then $\|\cdot\|$ is called a <u>norm</u>.

Examples

- $(\mathbb{R}^d, \|\cdot\|)$, d-dimensional real space with the Euclidean norm
- $L^1(\mu)$ (Norm: $\|f\|_{L^1(\mu)} = \int |f| \, d\mu$)
- $C_b(\mathbb{R}) = \{ f : \mathbb{R} \to \mathbb{R} \mid f$ continuous and bounded $\}$.

  Norm: supremum norm, $\|f\|_\infty := \sup_{x \in \mathbb{R}} |f(x)|$.

Notes

- Any norm gives rise to a metric, ρ, on X defined by

  $$\rho(x,y) := \|x-y\|$$

- Two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ are said to be <u>equivalent</u> if $\exists C \geq 1$ s.t.

  $$C^{-1}\|x\|_1 \leq \|x\|_2 \leq C\|x\|_1$$

- Any norm defines a topology called the <u>norm topology</u> (strong topology).

Convergence:

Def: Let $(X, \|\cdot\|)$ be a normed linear space. A series $\sum_{n=1}^{\infty} x_n$ is said to converge if

$$\lim_{N \to \infty} \left\| \sum_{n=1}^{N} x_n - x \right\| = 0$$

This is denoted $\sum_{n=1}^{\infty} x_n = X$.

The series, $\sum_{n=1}^{\infty} x_n$, is said to converge absolutely if

$$\sum_{n=1}^{\infty} \|x_n\| < \infty$$

Thm: A normed linear space, $(X, \|\cdot\|)$, is complete iff every absolutely convergent series in X converges to an element in X.

$\boxed{\text{Proof}}$

Banach Spaces.

Def: (Banach space)

If a normed linear space, $(X, \|\cdot\|)$, is complete (as a metric space) then $(X, \|\cdot\|)$ is called a **Banach Space**.

Examples:

1. $L^1(\mu)$ is a Banach Space
2. $L^p(\mu) := \{f : X \to \mathbb{R} \mid \int_X |f|^p < \infty\}$.

   with the norm $\|f\|_p := \left(\int_X |f|^p d\mu\right)^{1/p}$

   is a Banach space.
3. $C([0,1]) := \{f : [0,1] \to \mathbb{R} \mid f \text{ continuous}\}$.

   with the norm $\|f\|_\infty := \sup_{x \in [0,1]} |f(x)|$.

   is a Banach space

(4) $C_c(\mathbb{R}) := \{ f : \mathbb{R} \to \mathbb{R} \mid f \text{ continuous, compactly supported} \}$

w/ the sup norm

is a normed linear space, but not a Banach Space (On Pset 1).

Bases

Def: (Hamel Basis)

A subset, $\mathcal{B}$, of a linear space, X, is called a <u>Hamel Basis</u> for X if each $x \in X$ can be expressed as a unique, finite linear combination of elements in $\mathcal{B}$.

Lemma: Every linear space has a Hamel basis (Pset 1)

<u>Def:</u> (Schauder Basis)

A subset, $\{e_n\}_{n=1}^{\infty}$, of a normed linear space, $(X, \|\cdot\|)$, is called a <u>Schauder Basis</u> if for each $x \in X$ there exists a unique sequence, $\{x_n\}_{n=1}^{\infty} \subset \mathbb{R}$, s.t.

$$\sum_{n=1}^{\infty} x_n e_n = x$$

If the convergence is absolute, then $\{e_n\}_{n=1}^{\infty}$ is an <u>unconditional Schauder Basis</u>