Homework Problems
Math 525, Winter 2023
Due 11:00 pm, March 9, 2023

Instructions: Please write your solution to each problem on a separate page, and please include the full problem statement at the top of the page. All solutions must be written in legible handwriting or typed (in each case, the text should be of a reasonable size).

Your solutions to all problems should be written in complete sentences, with proper grammatical structure.

If your solutions are not typed, you must scan your written solutions and submit the digital copy. When submitting problems through LaTex, the LaTex source file (.tex) must be included in the submission.

1. * (Folland, Chapter 4, Section 6, Problem 63) Let $K \in C([0,1] \times [0,1])$. For $f \in C([0,1])$ define $Tf(x) = \int_0^1 K(x,y)f(y)dy$. Then $Tf \in C([0,1])$, and $\{Tf : \|f\|_\infty \leq 1\}$ is precompact in $C([0,1])$.

2. * (Folland, Chapter 4, Section 6, Problem 64) Let (X, ρ) be a metric space. A function $f \in C(X)$ is called Hölder continuous of exponent α, ($\alpha > 0$) if the quantity

$$N_\alpha(f) = \sup_{x \neq y} \frac{|f(x) - f(y)|}{\rho(x,y)^\alpha}$$

is finite. If X is compact, $\{f \in C(X) : \|f\|_u \leq 1 \text{ and } N_\alpha(f) \leq 1\}$ is compact in $C(X)$.

3. * Fix a function $\varphi \in C_c(\mathbb{R})$, $\varphi \not\equiv 0$, and consider the family of functions

$$\mathcal{F} = \{\varphi_n\}_{n=1}^\infty$$

where $\varphi_n(x) := \varphi(x+n)$.

   (a) Assume $1 \leq p < \infty$. Prove that for all $\varepsilon > 0$, there exists $\delta > 0$ such that if $|h| < \delta$ then

$$\|\tau_h f - f\|_p < \varepsilon$$

   for all $f \in \mathcal{F}$.

   (b) Show that $\mathcal{F}$ is not precompact.

4. * Let $1 \leq p < \infty$ and let $\mathcal{F} \subset L^p(\mathbb{R}^d)$ be a compact subset of $L^p(\mathbb{R}^d)$.

   (a) Prove that for all $\varepsilon > 0$, there exists $\delta > 0$ such that if $\|h\| < \delta$ then

$$\|\tau_h f - f\|_p < \varepsilon$$

   for all $f \in \mathcal{F}$.

   (b) Prove that for all $\varepsilon > 0$ there exists an $\Omega \subset \mathbb{R}^d$ bounded and open such that

$$\|f\|_{L^p(\mathbb{R}^d \setminus \Omega)} < \varepsilon$$

   for all $f \in \mathcal{F}$.

5. * (Folland, Chapter 6, Section 3, Problem 31) (Generalized Hölder inequality) Suppose $1 \leq p_j \leq \infty$ and $\sum_1^n p_j^{-1} = r^{-1} \leq 1$. If $f_j \in L^{p_j}$ for $j = 1, \ldots, n$, the $\prod_1^n f_j \in L^r$ and $\|\prod_1^n f_j\|_r \leq \prod_1^n \|f_j\|_{p_j}$.

6. (Folland, Chapter 6, Section 3, Problem 34) If f is absolutely continuous on $[\varepsilon, 1]$ for $0 < \varepsilon < 1$ and $\int_0^1 x\,|f'(x)|^p\,dx < \infty$, then $\lim_{x\to 0} f(x)$ exists (and is finite) if $p > 2$, $|f(x)|/|\log x|^{1/2} \to 0$ as $x \to 0$ if $p = 2$, and $|f(x)|/x^{1-(2/p)} \to 0$ as $x \to 0$ if $p < 2$.

7. * Prove that if p and q are conjugate exponents, and if $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$, then $f * g$ exists for every x, and $f * g$ is bounded and uniformly continuous.

8. Let $(X, \mathcal{M}, \mu)$ be a complete σ-finite measure space.

(a) Show that for $p \geq 2$ and $x \in [0,1]$

$$\left(\frac{1+x}{2}\right)^p + \left(\frac{1-x}{2}\right)^p \leq \frac{1}{2}(1+x^p)$$

(b) Prove that for $2 \leq p < \infty$, if $f, g \in L^p(\mu)$ then

$$\left\|\frac{f+g}{2}\right\|_p^p + \left\|\frac{f-g}{2}\right\|_p^p \leq \frac{1}{2}\left(\|f\|_p^p + \|g\|_p^p\right)$$

This is Clarkson inequality.

(c) Show that for $p \geq 2$, L^p is uniformly convex. Recall this means that given $\epsilon > 0$ there is $\delta = \delta(\epsilon) \in (0,1)$ with $\delta(\epsilon) \to 0$ as $\epsilon \to 0$ s so that whenever $\|f\|_p = \|g\|_p = 1$ then $\|f - g\|_p \geq \epsilon$ implies that $\left\|\frac{f+g}{2}\right\|_p \leq 1 - \delta$.

(d) Using the result above prove that for $2 \leq p < \infty$ the following statement holds. If $\{f_n\}_n \in L^p$ converges weakly to f and $\|f_n\|_p \to \|f\|_p$ then $\{f_n\}_n$ converges strongly to f, that is $\|f_n - f\|_p \to 0$ as $n \to \infty$.