Homework Problems
Math 525, Winter 2022
Due 11:00 pm, February 24, 2022

Instructions: Please write your solution to each problem on a separate page, and please include the full problem statement at the top of the page. All solutions must be written in legible handwriting or typed (in each case, the text should be of a reasonable size).

Your solutions to all problems should be written in complete sentences, with proper grammatical structure.

If your solutions are not typed, you must scan your written solutions and submit the digital copy. When submitting problems through LaTex, the LaTex source file (.tex) must be included in the submission.

1. * Assume $\Omega \subset \mathbb{R}^d$ and $m(\Omega) < \infty$.
   (a) Let $f \in L^\infty(\Omega)$. Prove that $\lim_{p\to\infty} \|f\|_p = \|f\|_\infty$.
   (b) Let $f \in \cap_{1\leq p<\infty} L^p(\Omega)$ and assume that there is a constant C such that
   $$\|f\|_p \leq C \quad \forall 1 \leq p < \infty.$$
   Prove that $f \in L^\infty(\Omega)$.
   (c) Construct an example of a function $f \in \cap_{1\leq p<\infty} L^p(\Omega)$ such that $f \notin L^\infty(\Omega)$ with $\Omega = (0, 1)$.
2. Assume $\Omega \subset \mathbb{R}^d$. Let $f : \Omega \to \mathbb{R}$ be a measurable function and let $1 \leq p \leq \infty$. Define the set
   $$C := \{u \in L^p(\Omega) : \quad u \geq f \quad \text{a.e. }\}.$$
   (a) Assume first that $1 \leq p < \infty$. Prove that C is convex and closed in the strong L^p topology. Deduce that C is weakly closed in L^p.
   (b) Taking $p = \infty$, prove that
   $$C = \left\{ u \in L^\infty(\Omega) : \begin{array}{l} \int u\varphi \geq \int f\varphi \quad \forall \varphi \in L^1(\Omega) \\ \text{with} \quad f\varphi \in L^1(\Omega) \quad \text{and } \varphi \geq 0 \quad \text{a.e.} \end{array} \right\}$$
   (c) Deduce that when $p = \infty$, C is weak* closed.
   (d) Let $f_1, f_2 \in L^\infty(\Omega)$ with $f_1 \leq f_2$ a.e. Prove that the set
   $$C = \{u \in L^\infty(\Omega) : \quad f_1 \leq u \leq f_2 \quad \text{a.e. }\}$$
   is compact in the weak* topology on $L^\infty(\Omega)$.
3. Let $\mathbb{T} := \mathbb{R}/\mathbb{Z}$, and let $(\mathcal{M}(\mathbb{T}), \|\cdot\|)$ represent the the Banach space of real-valued Radon measures on $\mathbb{T}$ with norm
   $$\|\mu\| := |\mu|(\mathbb{T}) = \sup\left\{ \sum_{i=1}^{\infty} |\mu(E_i)| \;:\; E_i \subset \mathbb{T} \text{ pairwise disjoint.} \right\}.$$
   (a) For $x \in \mathbb{T}$, let $T_x : \mathcal{M}(\mathbb{T}) \to \mathcal{M}(\mathbb{T})$ be the adjoint to the translation operator, $\tau_x f(y) := f(y - x)$. For a fixed $\mu \in \mathcal{M}(\mathbb{T})$, show that the function $f : \mathbb{T} \to \mathcal{M}(\mathbb{T})$ defined by
   $$f(x) := T_x\mu$$
   is continuous with respect to the weak-$*$ topology on $\mathcal{M}(\mathbb{T})$, but is not continuous with respect to the norm topology.
   (b) Let
   $$P := \{\mu \in \mathcal{M}(\mathbb{T}) \mid \mu(\mathbb{T}) = 1 \text{ and } \mu(E) \geq 0 \text{ for all } E \subset \mathbb{T}\}$$
   Show that P is convex and weak-$*$ compact.

(c) Show that for all $x \in \mathbb{T}$, there exists $\mu \in P$ such that

$$T_x(\mu) = \mu$$

4. * Let $\Omega \subset \mathbb{R}^d$ be open.

   (a) Let $u \in L^\infty(\Omega)$. Prove that there exists a sequence (u_n) in $C_c^\infty(\Omega)$ such that
      i. $\|u_n\|_\infty \leq \|u\|_\infty \quad \forall n$,
      ii. $u_n \to u$ a.e. on Ω,
      iii. u_n converges weak* to u in $L^\infty(\Omega)$.

   (b) Deduce that $C_c^\infty(\Omega)$ is dense in $L^\infty(\Omega)$ with respect to the weak* topology.

5. (Folland, Chapter 6, Section 1, Problem 2) Prove Theorem 6.8

6. (Folland, Chapter 6, Section 1, Problem 5) Suppose $0 < p < q \leq \infty$. Then $L^p \not\subset L^q$ iff X contains sets of arbitrarily small positive measure, and $L^q \not\subset L^p$ iff X contains sets of arbitrarily large finite measure. What about the case $q = \infty$?

7. * (Folland, Chapter 6, Section 1, Problem 9) If $\|f_n - f\|_p \to 0$ where $p < \infty$, then $f_n \to f$ in measure, and hence some subsequence converges to f a.e. On the other hand, if $f_n \to f$ in measure and $|f_n| \leq g \in L^p$ for all n where $p < \infty$, then $\|f_n - f\|_p \to 0$.

8. * (Folland, Chapter 6, Section 1, Problem 10) If $f_n, f \in L^p$ $(p < \infty)$ and $f_n \to f$ a.e., then $\|f_n - f\|_p \to 0$ iff $\|f_n\|_p \to \|f\|_p$.

9. (Folland, Chapter 6, Section 2, Problem 18) The self-duality of L^2 follows from Hilbert space theory, and this fact can be used to prove the Lebesgue-Radon-Nikodym theorem by the following argument due to von Neumann. Suppose that μ, ν are positive finite measures on $(X, \mathcal{M})$, and let $\lambda = \mu + \nu$.

   (a) The map $f \to \int f d\nu$ is a bounded linear functional on $L^2(\lambda)$, so $\int f d\nu = \int fg\, d\lambda$ for some $g \in L^2(\lambda)$. Equivalently, $\int f(1-g) d\nu = \int fg\, d\mu$ for $f \in L^2(\lambda)$.

   (b) $0 \leq g \leq 1$ λ-a.e., so we may assume $0 \leq g \leq 1$ everywhere.

   (c) Let $A = \{x : g(x) < 1\}, B = \{x : g(x) = 1\}$, and set $\nu_a(E) = \nu(A \cap E), \quad \nu_s(E) = \nu(B \cap E)$. Then $\nu_s \perp \mu$ and $d\nu_a = g(1-g)^{-1}\chi_A d\mu$.

10. * (Folland, Chapter 6, Section 2, Problem 20) Suppose $\sup_n \|f_n\|_p < \infty$ and $f_n \to f$ a.e.

    (a) If $1 < p < \infty$, then $f_n \to f$ weakly in L^p.

    (b) The result of (a) is false in general for $p = 1$. It is, however, true for $p = \infty$ if μ is σ-finite and weak convergence is replaced by weak* convergence.

11. (Folland, Chapter 6, Section 3, Problem 27) (Hilbert's Inequality) The operator $Tf(x) = \int_0^\infty (x + y)^{-1} f(y)\, dy$ satisfies $\|Tf\|_p \leq C_p \|f\|_p$ for $1 < p < \infty$, where $C_p = \int_0^\infty x^{-1/p}(x+1)^{-1} dx$. [For those who know about contour integrals: show that $C_p = \pi \csc(\pi/p)$.]

12. (Folland, Chapter 6, Section 5, Problem 43) Let H be the Hardy-Littlewood maximal operator on $\mathbb{R}$. Compute $H\chi_{(0,1)}$ explicitly. Show that it is in L^p for $p > 1$ and in weak L^1 but not in L^1, and that its L^p norm tends to ∞ like $(p-1)^{-1}$ as $p \to 1$, although $\left\|\chi_{(0,1)}\right\|_p = 1$ for all p.

13. * (Folland, Chapter 6, Section 5, Problem 45) If $0 < \alpha < n$, define an operator T_α on functions on $\mathbb{R}^n$ by

$$T_\alpha f(x) = \int |x - y|^{-\alpha} f(y) dy$$

    Then T_α is weak type $\left(1, (n-\alpha)^{-1}\right)$ and strong type (p, r) with respect to Lebesgue measure on $\mathbb{R}^n$ where $1 < p < n\alpha^{-1}$ and $r^{-1} = p^{-1} - \alpha n^{-1}$.