Homework Problems
Math 525, Winter 2023
Due 11:00 pm, February 9, 2023

Instructions: Please write your solution to each problem on a separate page, and please include the full problem statement at the top of the page. All solutions must be written in legible handwriting or typed (in each case, the text should be of a reasonable size).

Your solutions to all problems should be written in complete sentences, with proper grammatical structure.

If your solutions are not typed, you must scan your written solutions and submit the digital copy. When submitting problems through LaTex, the LaTex source file (.tex) must be included in the submission.

1. (Folland, Chapter 5, Section 4, Problem 47) Suppose that X and Y are Banach spaces.
   (a) If $\{T_n\}_{n=1}^{\infty} \subset \mathcal{L}(X, Y)$ and $T_n \to T$ weakly (or strongly), then $\sup_n \|T_n\| < \infty$.
   (b) Every weakly convergent sequence in X, and every weak*-convergent sequence in X^*, is bounded.
2. * (Folland, Chapter 5, Section 4, Problem 48) Suppose that X is a Banach Space
   (a) The norm-closed unit ball $B = \{x \in X : \|x\| \leq 1\}$ is also weakly closed.
   (b) If $E \subset X$ is bounded, so is its weak closure.
   (c) If $F \subset X^*$ is bounded, so is its weak* closure.
   (d) Every weak*-Cauchy sequence in X^* converges.
3. * (Folland, Chapter 5, Section 4, Problem 50) If X is a separable normed linear space, the weak* topology on the closed unit ball in X^* is second countable and hence metrizable.
4. Let $(X, \|\cdot\|)$ be a real Banach space. Show that X is a real Hilbert space if and only if for every $x, y \in X$, the norm satisfies the parallelogram law
$$\|x - y\|^2 + \|x + y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$
   (Hint: Define $\langle x, y\rangle := \frac{1}{2}(\|x + y\|^2 - \|x\|^2 - \|y\|^2)$.)
5. * (Folland, Chapter 5, Section 5, Problem 55) Let H be a Hilbert Space
   (a) (The polarization identity) For any $x, y \in \mathscr{X}$,
$$4\langle x, y\rangle = \|x + y\|^2 - \|x - y\|^2 + i\|x + iy\|^2 - i\|x - iy\|^2.$$
   (b) If H' is a Hilbert space, a linear map from H to H' is unitary iff it is isometric and surjective.
6. Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space and $T \in \mathcal{L}(H, H)$. Show that if T is self-adjoint (i.e. $T = T*$), then
$$\|T\| = \sup_{\substack{f \in H \\ \|f\|=1}} |\langle Tf, f\rangle|.$$
7. * (Folland, Chapter 5, Section 5, Problem 59) Every closed convex set, K, in a Hilbert space has a unique element of minimal norm
8. Let $\mathcal{H}$ be a separable Hilbert space. Let $A \in \mathcal{L}(\mathcal{H}, \mathcal{H})$. Let A^* be the adjoint of A. Show that for every pair $a, b \in \mathcal{H}$, the system of equations
$$\begin{cases} x + A^* y = a \\ Ax - y = b \end{cases}$$
   has a unique solution $x, y \in \mathcal{H}$ provided that the range of $I + A^*A$ is closed.

9. * Given an integer $n \geq 1$, write $n = k + 2^p$ where $p \geq 0$ and $k \geq 0$ are integers uniquely determined by the condition $k \leq 2^p - 1$. Consider the function defined on $(0, 1)$ by

$$\varphi_n(t) = \begin{cases} 2^{p/2}, & \text{if } k2^{-p} < t < \left(k + \frac{1}{2}\right) 2^{-p}, \\ -2^{p/2}, & \text{if } \left(k + \frac{1}{2}\right) 2^{-p} < t < (k+1)\, 2^{-p}, \\ 0 & \text{else.} \end{cases}$$

Set $\varphi_0 \equiv 1$, then show that $(\varphi_n)_{n=0}^{\infty}$ is an orthonormal basis for $L^2(0, 1)$.

10. Let P be the orthogonal projection associated with the closed subspace $\mathcal{S}$ of a Hilbert space $\mathcal{H}$, that is

$$P(f) = f \text{ if } f \in \mathcal{S} \quad \text{ and } \quad P(f) = 0 \text{ if } f \in \mathcal{S}^{\perp}.$$

   (a) Show that $P \circ P = P^2 = P$ and $P^* = P$.

   (b) Conversely if P is a bounded linear operator satisfying $P \circ P = P^2 = P$ and $P^* = P$, prove that P is the orthogonal projection for some closed subspace of $\mathcal{H}$.

11. (Prelim 2011) Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces and $T : \mathcal{X} \to \mathcal{Y}$ a one-to-one bounded linear map whose range $T(\mathcal{X})$ is closed in $\mathcal{Y}$. Show that for each bounded linear functional ϕ on $\mathcal{X}$ there is a bounded linear functional ψ on $\mathcal{Y}$ such that $\phi = \psi \circ T$, and there is a constant C (independent of ϕ) such that ψ can be chosen to satisfy $\|\psi\| \leq C\|\phi\|$. Here $\|\cdot\|$ denotes the operator norm.

12. (Folland, Chapter 5, Section 5, Problem 61) Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be σ-finite measure spaces such that $L^2(\mu)$ and $L^2(\nu)$ are separable. If $\{f_m\}$ and $\{g_n\}$ are orthonormal bases for $L^2(\mu)$ and $L^2(\nu)$, and $h_{mn}(x, y) = f_m(x)g_n(y)$, then $\{h_{mn}\}$ is an orthonormal basis for $L^2(\mu \times \nu)$.

13. * (Folland, Chapter 5, Section 5, Problem 64) Let H be a separable infinite-dimensional Hilbert space with orthonormal basis $\{u_n\}_{n=1}^{\infty}$.

   (a) Define $L_k \in \mathcal{L}(H, H)$ by $L_k\left(\sum a_n u_n\right) = a_n u_{n-k}$. Then $L_k \to 0$ in the strong operator topology but not in the norm topology.

   (b) Define $R_k \in \mathcal{L}(H, H)$ by $R_k\left(\sum a_n u_n\right) = \sum a_n u_{n+k}$. Then $R_k \to 0$ in the weak operator topology but not in the strong operator topology.

   (c) $R_k L_k \to 0$ in the strong operator topology, but $L_k R_k = I$ for all k.

14. (Prelim 2017) Suppose $\mathcal{H}$ is a Hilbert space and $y_n \in \mathcal{H}$ are such that the sequence $\langle x, y_n \rangle$ converges, as $n \to \infty$, for every $x \in \mathcal{H}$. Prove that there exists $y \in \mathcal{H}$ such that $\lim_{n\to\infty} \langle x, y_n \rangle = \langle x, y \rangle$ for every $x \in \mathcal{H}$.