Homework Problems
Math 525, Winter 2023
Due 11:00 pm, January 26, 2023

Instructions: Please write your solution to each problem on a separate page, and please include the full problem statement at the top of the page. All solutions must be written in legible handwriting or typed (in each case, the text should be of a reasonable size).

Your solutions to all problems should be written in complete sentences, with proper grammatical structure.

If your solutions are not typed, you must scan your written solutions and submit the digital copy. When submitting problems through LaTex, the LaTex source file (.tex) must be included in the submission.

1. Show that every nonempty weakly open subset of an infinite dimensional normed linear space is unbounded with respect to the norm.

2. * (Folland, Chapter 5, Section 1, Problem 12) Let X be anormed vector space and M a proper closed subspace of X.
   (a) $\|x+M\| = \inf\{\|x+y\| : y \in M\}$ is a norm on X/M.
   (b) For any $\varepsilon > 0$ there exists $x \in X$ such that $\|x\| = 1$ and $\|x+M\| \geq 1-\varepsilon$.
   (c) The projection map $\pi(x) = x + M$ from X to X/M has norm 1.
   (d) If X is complete, so is X/M.
   (e) The topology defined by the quotient norm is the quaotient topology as defined in Exercise 28 in Chapter 4, Section 2.

3. * (Folland, Chapter 5, Section 1, Problem 16) Let $(X, \mathcal{M}, \mu)$ be a measure space, $\mathscr{Y}$ a separable Banach space, and $L_{\mathscr{Y}}$ the space of all $(\mathcal{M}, \mathcal{B}_{\mathscr{Y}})$-measurable maps from X to $\mathscr{Y}$, and $F_{\mathscr{Y}}$ the set of maps $f : X \to \mathscr{Y}$ of the form

$$f(x) = \sum_{j=1}^{n} \chi_{E_j}(x) y_j$$

where $n \in \mathbb{N}$, $y_j \in \mathscr{Y}$, $E_j \in \mathcal{M}$, and $\mu(E_j) < \infty$. If $f \in L_{\mathscr{Y}}$, since $y \mapsto \|y\|$ is continuous, $x \mapsto \|f(x)\|$ is $(\mathcal{M}, \mathcal{B}_{\mathbb{R}})$-measurable and we define

$$\|f\|_1 := \int \|f(x)\| \, d\mu(x).$$

Finally, let $L^1_{\mathscr{Y}} = \{f \in L_{\mathscr{Y}} : \|f\|_1 < \infty\}$.

   (a) $L_{\mathscr{Y}}$ is a vector space, $F_{\mathscr{Y}}$ and $L^1_{\mathscr{Y}}$ are subspaces of it, $F_{\mathscr{Y}} \subset L^1_{\mathscr{Y}}$, and $\|\cdot\|_1$ is a seminorm on $L^1_{\mathscr{Y}}$ that becomes a norm if we identify two functions that are equal μ-a.e.
   (b) Let $\{y_n\}_1^\infty$ be a countable dense set in $\mathscr{Y}$. Given $\varepsilon > 0$, let $B_n^\varepsilon = \{y \in \mathscr{Y} : \|y - y_n\| < \varepsilon \|y_n\|\}$. Then $\bigcup_1^\infty B_n^\varepsilon \supset (\mathscr{Y} \setminus \{0\})$.
   (c) If $f \in L^1$, there is a sequence $\{h_n\} \subset F$ with $h_n \to f$ a.e. and $\|h_n - f\|_1 \to 0$. [With notation as in (b) , let $A_{nj} = B_n^{1/j} \setminus \cup_{m=1}^{n-1} B_m^{1/j}$ and $E_{nj} = f^{-1}(A_{nj})$, and consider $g_j = \sum_1^\infty y_n \chi_{E_{nj}}$.]
   (d) There is a unique linear map $\int : L^1_{\mathscr{Y}} \to \mathscr{Y}$ such that $\int y\chi_E = \mu(E) y$ for $y \in \mathscr{Y}$ and $E \in \mathscr{M}$ $(\mu(E) < \infty)$, and $\left\|\int f\right\| \leq \|f\|_1$.
   (e) The dominated convergence theorem: If $\{f_n\}$ is a sequence in $L^1_{\mathscr{Y}}$ and there exists $g \in L^1$ such that $\|f_n(x)\| \leq g(x)$ for a.e. x, and $f_n \to f$ a.e., then $\int f_n \to \int f$.
   (f) If $\mathscr{Z}$ is a separable Banach space, $T \in L(\mathscr{Y}, \mathscr{Z})$, and $f \in L^1_{\mathscr{Y}}$, then $T \circ f \in L^1_{\mathscr{Z}}$ and $\int T \circ f = T(\int f)$.

4. (Prelim 2014) Let $\mathcal{B}$ be a real Banach space. Denote by $\|\cdot\|$ the norm on $\mathcal{B}$. Suppose that $\mathcal{B}$ satisfies the best approximation property, that is: Given a closed subspace $\mathcal{M}$ of $\mathcal{B}$ and given $x \in \mathcal{B}$ there exists $y_x \in \mathcal{M}$ such that $\|x - y_x\| = \inf_{y\in\mathcal{M}} \|y - x\| = \text{dist}(x, \mathcal{M})$.

   Prove that for every bounded linear function $f \in \mathcal{B}^*$ there exists $z \in \mathcal{B}$ such that $z \neq 0$ and $f(z) = \|f\|\|z\|$.

5. (Folland, Chapter 5, Section 2, Problem 17) A linear functional f on a normed vector space $\mathscr{X}$ is bounded iff $f^{-1}(\{0\})$ is closed.

6. * (Folland, Chapter 5, Section 2, Problem 19) Let X be an infinite-dimensional normed vector space.

   (a) There is a sequence $\{x_j\}$ in X such that $\|x_j\| = 1$ for all j and $\|x_j - x_k\| \geq \frac{1}{2}$ for $j \neq k$.

   (b) X is not locally compact.

7. (Folland, Chapter 5, Section 2, Problem 21) If X and Y are normed vector spaces, define $\alpha : X^* \times Y^* \to (X \times Y)^*$ by $\alpha(f, g)(x, y) = f(x) + g(y)$. Then α is an isomorphism which is isometric if we use the norm $\|(x, y)\| = \max(\|x\|, \|y\|)$ on $X \times Y$ and the corresponding operator norm on $(X \times Y)^*$, and the norm $\|(f, g)\| = \|f\| + \|g\|$ on $X^* \times Y^*$.

8. * (Folland, Chapter 5, Section 2, Problem 23) Suppose $\mathscr{X}$ is a Banach space. If $\mathscr{M}$ is a closed subspace of $\mathscr{X}$ and $\mathscr{N}$ is a closed subspace of $\mathscr{X}^*$, let $\mathscr{M}^0 = \{f \in \mathscr{X}^* : f|_{\mathscr{M}} = 0\}$ and $\mathscr{N}^\perp = \{x \in \mathscr{X} : f(x) = 0$ for all $f \in \mathscr{N}\}$. (Thus, if we identify $\mathscr{X}$ with its image in $\mathscr{X}^{**}$, $\mathscr{N}^\perp = \mathscr{N}^0 \cap \mathscr{X}$.)

   (a) $\mathscr{M}^0$ and $\mathscr{N}^\perp$ are closed subspaces of $\mathscr{X}^*$ and $\mathscr{X}$ respectively.

   (b) $\left(\mathscr{M}^0\right)^\perp = \mathscr{M}$ and $\left(\mathscr{N}^\perp\right)^0 \supset \mathscr{N}$. If $\mathscr{X}$ is reflexive, $\left(\mathscr{N}^\perp\right)^0 = \mathscr{N}$.

   (c) Let $\pi : \mathscr{X} \to \mathscr{X}/\mathscr{M}$ be the canonical projection, and define $\alpha : (\mathscr{X}/\mathscr{M})^* \to \mathscr{X}^*$ by $\alpha(f) = f \circ \pi$. Then α is an isometric isomorphism from $(\mathscr{X}/\mathscr{M})^*$ onto $\mathscr{M}^0$, where $\mathscr{X}/\mathscr{M}$ has the quotient norm (cf. Exercise 11).

   (d) Define $\beta : \mathscr{X}^* \to \mathscr{M}^*$ by $\beta(f) = (f|_{\mathscr{M}})$. By Exercise 14, β induces a map $\bar{\beta} : \mathscr{X}^*/\mathscr{M}^0 \to \mathscr{M}^*$. This $\bar{\beta}$ is an isometric isomorphism.

9. * (Folland, Chapter 5, Section 2, Problem 25) If X is a Banach space and X^* is separable, then X is separable.

10. (Folland, Chapter 5, Section 3, Problem 27) There exist meager subsets of $\mathbb{R}$ whose complements have Lebesgue measure zero.

11. (Folland, Chapter 5, Section 3, Problem 30) Let $Y = C([0, 1])$ and $X = C^1([0, 1])$, both equipped with the uniform norm.

    (a) X is not complete.

    (b) The map $(d/dx) : X \to Y$ is closed but not bounded.

12. (Folland, Chapter 5, Section 3, Problem 31) Let X, Y be Banach spaces and let $S : X \to Y$ be an unbounded linear map. Let $\Gamma(S)$ be the graph of S, a subspace of $X \times Y$.

    (a) $\Gamma(S)$ is not complete.

    (b) Define $T : X \to \Gamma(S)$ by $Tx = (x, Sx)$. Then T is closed but unbounded.

    (c) $T^{-1} : \Gamma(S) \to X$ is bounded and surjective but not open.

13. (Folland, Chapter 5, Section 3, Problem 38) Let X and Y be Banach spaces, and let $\{T_n\}$ be a sequence in $L(X, Y)$ such that $\lim T_n x$ exists for every $x \in X$. If $Tx = \lim T_n x$, then $T \in L(X, Y)$.

14. Prove that if (X, d) is a complete metric space and $F : X \to \mathbb{R}$ is a pointwise limit of continuous functions on X then the set, E, of points of continuity of F is dense in X.