Homework Problems
Math 525, Winter 2023
Due 11:00 pm, January 12, 2023

Instructions: Please write your solution to each problem on a separate page, and please include the full problem statement at the top of the page. All solutions must be written in legible handwriting or typed (in each case, the text should be of a reasonable size).

Your solutions to all problems should be written in complete sentences, with proper grammatical structure.

If your solutions are not typed, you must scan your written solutions and submit the digital copy. When submitting problems through LaTex, the LaTex source file (.tex) must be included in the submission.

1. Let $(X, \|\cdot\|)$ be a finite-dimensional normed linear space.
   (a) Show that every linear functional on X is continuous.
   (b) Let dim $X = n$. Let $\{e_1, \ldots, e_n\}$ be a basis for X. For $1 \leq i \leq n$, define $\psi_i \in X^*$ by $\psi_i(x) = x_i$ for $x = x_i e_i + \cdots + x_n e_n \in X$. Show that $\{\psi_1, \ldots, \psi_n\}$ is a basis for X^*. Thus dim $X^* = n$.
   (c) Show that the natural embedding, $J : X \to X^{**}$, is a bijection.

2. * (Folland, Chapter 5, Section 1, Problem 6) Let X be a finite-dimensional vector space. Let $\{e_1, \ldots, e_n\}$ be a basis for X and define

$$\left\| \sum_{j=1}^{n} a_j e_j \right\|_1 := \sum_{j=1}^{n} |a_j|$$

   (a) $\|\cdot\|_1$ is a norm on X.
   (b) The map $(a_1, ..., a_n) \mapsto \sum_{j=1}^{n} a_j e_j$ is continuous from K^n with the Euclidean topology to X with the topology defined by $\|\cdot\|_1$.
   (c) The set $\{x \in X : \|x\|_1 = 1\}$ is compact in the topology defined by $\|\cdot\|_1$.
   (d) All norms on X are equivalent.

3. Show that every nonempty weakly open subset of an infinite dimensional normed linear space is unbounded with respect to the norm.

4. Let X be a linear space. Apply Zorn's Lemma to conclude that X has a Hamel basis.

5. * (Folland, Chapter 5, Section 1, Problem 7) Let X be a Banach space.
   (a) If $T \in L(X, X)$ and $\|I - T\| < 1$ (where I is the identity operator), then T is invertible; in fact, the series $\sum_{n=0}^{\infty}(I - T)^n$ converges in $L(X, X)$ to T^{-1}
   (b) If $T \in L(X, X)$ is invertible and $\|S - T\| < \|T^{-1}\|^{-1}$, then S is invertible. Thus the set of invertible operators is open in $L(X, X)$.

6. * Let X be a topological space. Consider the following class of real-valued functions on X:

$$C_c(X) := \{f : X \to \mathbb{R} \mid f \text{ is continuous and has compact support}\}.$$

   Furthermore, define the function, $\|\cdot\|_\infty : C_c(X) \to \mathbb{R}$, by

$$\|f\|_\infty := \sup_{x \in X} |f(x)|.$$

   Show that $(C_c(X), \|\cdot\|_\infty)$ is a normed linear space. Provide an example that demonstrates that $(C_c(X), \|\cdot\|_\infty)$ is not necessarily a Banach space.

7. A topological space, X, is a Hausdorff space if for every pair $x, y \in X$ such that $x \neq y$, there exists two open sets, U and V, such that $x \in U$, $y \in V$, and $U \cap V = \emptyset$. A topological space, X, is locally compact if every point of X has an open neighborhood whose closure is compact. Let X be a locally compact, Hausdorff space. Let $F \subset V \subsetneq X$ where F is compact and V is open.

   (a) Prove that there exists an open set $U \subset X$ (with compact closure) satisfying
   $$F \subset U \subset \overline{U} \subset V.$$

   (b) Prove that there exists $f \in C_c(X)$ such that
   $$\chi_F(x) \leq f(x) \leq \chi_V(x)$$
   for all $x \in X$.

8. * Show that $C_c(\mathbb{R})$ is dense in $L^1(\mathbb{R})$.

9. * (Folland, Chapter 5, Section 1, Problem 9) Let $C^k([0,1])$ be the space of functions on $[0,1]$ possessing continuous derivatives up to order k on $[0,1]$, including one-sided derivatives at the endpoints.

   (a) If $f \in C([0,1])$, then $f \in C^k([0,1])$ iff f is k times continuously differentiable on $(0,1)$ and $\lim_{x \to 0^+} f^{(j)}(x)$ and $\lim_{x \to 1^-} f^{(j)}(x)$ exists $j \leq k$.

   (b) $\|f\| := \sum_{j=0}^{k} \|f^{(j)}\|_\infty$ is a norm on $C^k([0,1])$ that makes $C^k([0,1])$ into a Banach space.

10. (Folland, Chapter 5, Section 1, Problem 10) Let $L_k^1([0,1])$ be the space of all $f \in C^{k-1}[0,1]$ such that $f^{(k-1)}$ is absoutely continuous on $[0,1]$. Then $\|f\| := \sum_{j=0}^{k} \int_0^1 |f^{(j)}(x)|\, dm(x)$ is a norm on $L_k^1([0,1])$ that makes $L_k^1([0,1])$ into a Banach space.

11. * (Folland, Chapter 5, Section 1, Problem 11) If $0 < \alpha \leq 1$, let $\Lambda_\alpha([0,1])$ be the space of Hölder continuous functions of exponent α on $[0,1]$. That is, $f \in \Lambda_\alpha([0,1])$ iff $\|f\|_{\Lambda_\alpha} < \infty$, where
    $$\|f\|_{\Lambda_\alpha} := |f(0)| + \sup_{\substack{x,y \in [0,1] \\ x \neq y}} \frac{|f(x) - f(y)|}{|x-y|^\alpha}.$$

    (a) $\|\cdot\|_{\Lambda_\alpha}$ is a norm that makes $\Lambda_\alpha([0,1])$ into a Banach space.

    (b) Let $\lambda_\alpha([0,1])$ be the set of all $f \in \Lambda_\alpha([0,1])$ such that
    $$\frac{|f(x) - f(y)|}{|x-y|^\alpha} \to 0 \text{ as } x \to y, \text{ for all } y \in [0,1].$$
    If $\alpha < 1$, $\lambda_\alpha([0,1])$ is an infinite-dimensional closed subspace of $\Lambda_\alpha([0,1])$. If $\alpha = 1$, $\lambda_\alpha([0,1])$ contains only constant functions.

12. (Folland, Chapter 5, Section 1, Problem 15) Suppose that X and Y are normed vector spaces and $T \in L(X,Y)$. Let $N(T) = \{x \in X : Tx = 0\}$.

    (a) $N(T)$ is a closed subspace of X.

    (b) There is a unique $S \in L(X/N(T), Y)$ such that $T = S \circ \pi$ where $\pi : X \to X/N(T)$ is the projection. Moreover, $\|S\| = \|T\|$.