Prop: Let (X, ρ) be a metric space, μ be a Borel measure on X. Assume

$$\mu(B(x,r)) < \infty \qquad \forall x \in X, \ \forall r > 0.$$

Then

(*) For any $E \in \mathcal{B}_X$, $\forall \varepsilon > 0$ $\exists$ open set G and a closed set F such that

$$F \subset E \subset G \quad \text{and}$$

$$\mu(G \setminus E) < \varepsilon \quad \text{and} \quad \mu(E \setminus F) < \varepsilon.$$

A Borel measure, μ, satisfying (*) is called a Borel regular measure.

Proof of Prop

Consider the collection of sets.

$$\mathcal{C} := \left\{ E \in \mathcal{B}_X \;\middle|\; \begin{array}{l} \text{for all } \varepsilon > 0 \ \ \exists \ \overset{\text{open}}{G_\varepsilon}, \overset{\text{closed.}}{F_\varepsilon} \text{ s.t. } F_\varepsilon \subset E \subset G_\varepsilon \\ \mu(G_\varepsilon \setminus E) < \varepsilon \text{ and } \mu(E \setminus F_\varepsilon) < \varepsilon \end{array} \right\}$$

(1) Show that $\mathcal{C} \supset \mathcal{B}_X$

(a) Show that $\mathcal{C}$ is a σ-algebra

(b) Show that $\mathcal{C}$ contains open sets.

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(a) $\mathcal{C}$ is a σ-algebra

- $\phi, X \in \mathcal{C}$
- $E \in \mathcal{C} \Rightarrow E^c \in \mathcal{C}$.

Note : $G_\varepsilon \setminus E = E^c \setminus G_\varepsilon^c$

- Let $\{E_k\}_{k=1}^\infty \subset \mathcal{C}$, $\varepsilon > 0$

and $\overset{\text{closed}}{F_k} \subset E_k \subset \overset{\text{open}}{G_k}$

with $\mu(G_k \setminus E_k) < \frac{\varepsilon}{2^k}$, $\mu(E_k \setminus F_k) < \frac{\varepsilon}{2^k}$

Let $G := \bigcup_{k=1}^\infty G_k$

then $\mu(\bigcup G_k \setminus \bigcup E_k)$

$\leq \sum \mu(G_k \setminus E_k) < \varepsilon$.

Let $F := \bigcup_{k=1}^\infty F_k$

then $\mu(\bigcup E_k \setminus \bigcup F_k)$

$\leq \sum \mu(E_k \setminus F_k) < \varepsilon$

but F may not be closed.

Measurable Functions

Let $f : X \to Y$ and

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$f^{-1}: P(Y) \to P(X)$ be defined by

$f^{-1}(E) = \{x \in X \mid f(x) = y \text{ for some } y \in E\}$

Recall: ① $f^{-1}\left(\bigcup_\alpha E_\alpha\right) = \bigcup_\alpha f^{-1}(E_\alpha)$

② $f^{-1}\left(\bigcap_\alpha E_\alpha\right) = \bigcap_\alpha f^{-1}(E_\alpha)$

③ $f^{-1}(E^c) = [f^{-1}(E)]^c$

Which implies the following lemma

Lemma: If N is a σ-algebra in Y, then $\{f^{-1}(E) \mid E \in N\}$ is a σ-algebra in X.

Def: (Measurable Functions)

Let (M, X) and (N, Y) be measurable spaces

a function $f: X \to Y$ is (M, N) measurable

if $f^{-1}(E) \in M$ for all $E \in N$.

Note: If $f: (X, M) \to (Y, N)$ is (M,N)-measurable and $g: (Y, N) \to (Z, O)$ is (N, O)-measurable

then $g \circ f: (X, M) \to (Z, O)$ is (M, O)-measurable.

Examples

- $f: (X, 2^X) \to (Y, N)$ is measurable

- $f : \mathbb{R} \to \mathbb{R}$ continuous

  is $(\mathcal{B}_{\mathbb{R}}, \mathcal{B}_{\mathbb{R}})$ measurable.