Lemma: Let μ be a metric outer measure on (X, e), then all closed sets are μ-measurable, thus so are all open sets, and μ is Borel.

Measures

Def: Let $\mathcal{M} \subset \mathcal{P}(X)$ be a σ-algebra.

A measure is a function, $\mu : \mathcal{M} \to [0, \infty]$, satisfying

i.) $\mu(\emptyset) = 0$

ii.) If $\{E_i\}_{i=1}^{\infty} \subset \mathcal{M}$ is a disjoint collection of sets, then

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu(E_i)$$

- The triple $(X, \mathcal{M}, \mu)$ is called a measure space
- $(X, \mathcal{M}, \mu)$ is a complete measure space if $F \in \mathcal{M}$, $\mu(F) = 0$ and $E \subset F$ implies $E \in \mathcal{M}$.
- $(X, \mathcal{M}, \mu)$ is σ-finite if $\exists \{X_j\}_{j=1}^{\infty} \subset \mathcal{M}$ such that $X = \bigcup_{j=1}^{\infty} X_j$ and $\mu(X_j) < \infty$ for all j.

Ex: Outer measure restricted to measurable sets

Thm: Given an outer measure μ_* on $X \neq \phi$, the collection, $\mathcal{M}$, of μ_*-measurable sets is a σ-algebra. Moreover, $\mu := \mu_*|_{\mathcal{M}}$ is a measure and

$$(X, \mathcal{M}, \mu) \text{ is a complete measure space.}$$

Ex: For the Lebesgue outer measure, m_*, on $\mathbb{R}^d$
Denote the m_*-measurable sets by

$$\mathcal{L} := \{E \subset \mathbb{R}^d \mid E \text{ is } m_*\text{-measurable}\}.$$

Define

$$m := m_*|_{\mathcal{L}} : \mathcal{L} \to [0, \infty] \quad \text{is a measure}$$

and $(\mathbb{R}^d, \mathcal{L}, m)$ is a complete measure space known as the Lebesgue measure space where m is the Lebesgue measure.

Note: Since m_* is a metric outer measure,

$$\mathcal{B}_{\mathbb{R}^d} \subset \mathcal{L}.$$

Q: Is it true that $\mathcal{L} \subset \mathcal{B}_{\mathbb{R}^d}$? No HW 2

Prop: If μ_* on (X, ρ) is a metric outer measure then $\mu_*|_{\mathcal{B}_X}$ is a measure and if $\mathcal{M}$ is the collection of μ_*-measurable sets, then

$\mu = \mu_* |_{\mathcal{M}}$ is a complete measure and

$\mathcal{B}_X \subset \mathcal{M}$.

Prop: Let (X, ρ) be a metric space, μ be a Borel measure on X. Assume

$$\mu(B(x,r)) < \infty \qquad \forall x \in X, \ \forall r > 0.$$

Then

(*) For any $E \in \mathcal{B}_X$, $\forall \varepsilon > 0$ $\exists$ open set G and a closed set F such that

$$F \subset E \subset G \quad \text{and}$$

$$\mu(G \setminus E) < \varepsilon \quad \text{and} \quad \mu(E \setminus F) < \varepsilon.$$

A Borel measure, μ, satisfying (*) is called a Borel regular measure.

Examples

① $(\mathbb{R}^d, \mathcal{L}, m)$

② Let $\delta_{\{x\}}(E) := \begin{cases} 1 & , x \in E \\ 0 & , x \notin E \end{cases}$

Consider

$$\left(\mathbb{R}^d, \mathcal{L}, \sum_{i=1}^{\infty} \delta_{\{x_i\}}\right)$$

a.) if $\lim_{i \to \infty} x_i = \infty$ ✓

b.) if $\{x_i\}_{i=1}^{\infty}$ is bounded then

$\exists B(x,r)$ s.t. $\mu(B(x,r)) = \sum \delta_{\{x_i\}}(B(x,r)) = \infty$.

③ $(\mathbb{R}, \mathcal{B}_{\mathbb{R}}, \nu)$

where $\nu(E) := \int_E \frac{dx}{|x|}$ is a counter example.

<u>Proof of Prop</u>

Consider the collection of sets.

$$\mathcal{C} := \left\{ E \in \mathcal{B}_X \;\middle|\; \begin{array}{l} \text{for all } \varepsilon > 0 \;\; \exists \overset{\text{open}}{G_\varepsilon}, \overset{\text{closed.}}{F_\varepsilon} \text{ s.t. } F_\varepsilon \subset E \subset G_\varepsilon \\ \mu(G_\varepsilon \setminus E) < \varepsilon \text{ and } \mu(E \setminus F_\varepsilon) < \varepsilon \end{array} \right\}$$

① Show that $\mathcal{C} \supset \mathcal{B}_X$

ⓐ Show that $\mathcal{C}$ is a σ-algebra

ⓑ Show that $\mathcal{C}$ contains open sets.