Def.

(1) If $\mathcal{E} \subset \mathcal{P}(X)$ is a non-empty collection of subsets of X, the <u>σ-algebra generated by $\mathcal{E}$</u>, denote $\sigma(\mathcal{E})$ is the smallest σ-algebra containing $\mathcal{E}$, i.e. $\sigma(\mathcal{E}) = \bigcap_{\substack{\mathcal{M} \text{ σ-algebra} \\ \mathcal{E} \subset \mathcal{M}}} \mathcal{M}$.

(2) Let (X, ρ) be a metric space. The <u>Borel σ-Algebra</u> of X is the smallest σ-algebra containing the open subsets of X. Denoted $\mathcal{B}_X$

(3) An outer measure, μ, is called <u>Borel</u>, if each Borel set is μ-measurable.

<u>Example</u>: (Dyadic cubes in $\mathbb{R}^d$).

Let R, denote a closed rectangle in $\mathbb{R}^d$

$$R = [a_1, b_1] \times \cdots \times [a_d, b_d]$$

$$|R| := (\text{volume of } R) = (b_1 - a_1) \cdot (b_2 - a_2) \cdot \ldots \cdot (b_d - a_d).$$

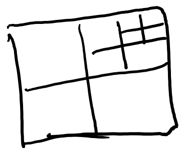
A <u>cube</u> is a rectangle satisfying $(b_i - a_i) = \ell$ for all $i = 1, \ldots, d$.

<u>Dyadic cubes.</u>

For each $m \in \mathbb{N}$, define

[illegible]

$$\mathcal{D}_m = \left\{ [k_1 2^{-m}, (k_1+1)2^{-m}] \times [k_2 2^{-m}, (k_2+1)2^{-m}] \times \cdots \times [k_d 2^{-m}, (k_d+1)2^{-m}] \;\middle|\; k_j \in \mathbb{Z} \text{ for all } j = 1, \ldots, d \right\}.$$



Let $\mathcal{D} = \bigcup_{m \in \mathbb{N}} \mathcal{D}_m$

Proposition: $\sigma(\mathcal{D}) = \mathcal{B}_{\mathbb{R}^d}$

Pf: we want to show $\sigma(\mathcal{D}) \subset \mathcal{B}_{\mathbb{R}^d}$ and

$\sigma(\mathcal{D}) \supset \mathcal{B}_{\mathbb{R}^d}$.

First direction: $\sigma(\mathcal{D}) \subset \mathcal{B}_{\mathbb{R}^d}$

It suffices to show that $\mathcal{B}_{\mathbb{R}^d}$ is a σ-algebra containing $\mathcal{D}$

Second direction: $\sigma(\mathcal{D}) \supset \mathcal{B}_{\mathbb{R}^d}$

Thm: Every open set in $\mathbb{R}^d$ can be written as a countable union of dyadic cubes. HW 2

$\square$

Def: (Lebesgue Outer measure) Recall from HW.

Let $E \subset \mathbb{R}^d$. Define

$$m_*(E) := \inf \left\{ \sum_{j=1}^{\infty} |Q_j| : E \subset \bigcup_{j=1}^{\infty} Q_j, \; Q_j \text{ closed cubes} \right\}.$$

Ex: ① $m_*(\{x\}) = 0$

② $m_*(R) = |R|$ when R is a cube (HW 1)

(2) $m_*(R) = |R|$ when R is a cube (HW 1)

(3) If C is the cantor set, then

$$m_*(C) = 0$$

Lemma: m_* is translation invariant. I.e. for any $E \subset \mathbb{R}^d$,

$$m_*(E+x) = m_*(E)$$

This implies that our construction of a nonmeasurable set in $\mathbb{R}$ is not m_*-measurable on $\mathbb{R}$.

Def: (Metric Outer Measure)

Let (X, ρ) be a metric space, an outer measure μ on X is a metric outer measure if

$$\mu(A \cup B) = \mu(A) + \mu(B)$$

whenever $d(A,B) := \inf_{\substack{a \in A \\ b \in B}} \rho(a,b) > 0$.

Ex: m_* is a metric outer measure (HW 1)

Lemma: Let μ be a metric outer measure on (X, ρ), then all closed sets are μ-measurable, thus so are all open sets, and μ is Borel.

Before we prove this lemma, we have the following corollary which follows from HW7

Corollary All closed and open sets in $\mathbb{R}^d$ are m_*-measurable and m_* is Borel.

Proof of Lemma

Let $F \subset X$ be closed, let $E \subset X$.

If $\mu(E) = +\infty$ then

$$\mu(E) \geq \mu(F \cap E) + \mu(F^c \cap E). \quad \checkmark$$

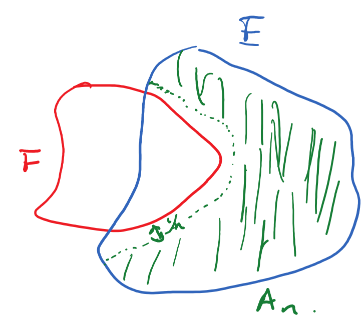
If $\mu(E) < \infty$ then

For $n \in \mathbb{N}$, let

$$A_n := \{x \in F^c \cap E \mid d(x, F) \geq \tfrac{1}{n}\}.$$

here $d(x,F) := \inf_{y \in F} \rho(x,y)$

Picture



Since μ is a metric outer measure,

$$\mu(E) \geq \mu(E \cap F \cup A_n)$$
$$= \mu(E \cap F) + \mu(A_n)$$

$$\Rightarrow \quad \mu(E) \geq \mu(E \cap F) + \lim_{n\to\infty} \mu(A_n).$$ Therefore, it suffices to show that

$$\lim_{n\to\infty} \mu(A_n) = \mu(E \cap F^c).$$