<u>Thm:</u> Let $\{A_k\}_{k=1}^{\infty}$ be a sequence of μ-measurable sets.

i.) The sets $\bigcup_{k=1}^{\infty} A_k$ and $\bigcap_{k=1}^{\infty} A_k$ are μ-measurable

ii.) If the sets $\{A_k\}_{k=1}^{\infty}$ are disjoint, then

$$\mu\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} \mu(A_k)$$

iii.) If $A_1 \subset A_2 \subset \cdots A_k \subset A_{k+1} \subset \cdots$ then

$$\lim_{k\to\infty} \mu(A_k) = \mu\left(\bigcup_{k=1}^{\infty} A_k\right)$$

iv) If $A_1 \supset A_2 \supset \cdots \supset A_k \supset A_{k+1} \supset \cdots$ and $\mu(A_1) < \infty$,

then $\lim_{k\to\infty} \mu(A_k) = \mu\left(\bigcap_{k=1}^{\infty} A_k\right)$.

<u>Proof:</u>

(i.) + (ii) First, consider two measurable sets

A_1 and A_2

Let's show that $A_1 \cup A_2$ is measurable

We need to show that for any $B \subset X$,

$$\mu(B) \geq \mu(B \cap (A_1 \cup A_2)) + \mu(B \cap (A_1 \cup A_2)^c).$$

$$= \mu\big((B\cap A_1)\cup(B\cap A_2)\big) + \mu(B\cap A_1^c\cap A_2^c)$$

Note that

$$\mu\big((B\cap A_1)\cup(B\cap A_2)\big) \le \mu(B\cap A_1) + \mu(B\cap A_2)$$

$$\Rightarrow \mu\big((B\cap A_1)\cup(B\cap A_2)\big) + \mu(B\cap A_1^c\cap A_2^c)$$

$$\le \mu(B\cap A_1) + \mu(B\cap A_2) + \mu(B\cap A_1^c \cap A_2^c).$$

It would be nice if $\mu(B\cap A_2) + \mu(B\cap A_1^c\cap A_2^c) \le \mu(B\cap A_1^c)$.

Not true: Let μ = counting measure, $B = \{x_1, \dots, x_{10}\} = X$

$A_2 = B$, $A_1 = \{x_1, \dots, x_5\}$

We need a more efficient decomposition of $B\cap(A_1\cup A_2)$.

Note: $B\cap(A_1\cup A_2) = (B\cap A_1)\cup(B\cap A_2\setminus A_1) = (B\cap A_1)\cup(B\cap A_2\cap A_1^c)$.

Now

$$\mu(B\cap(A_1\cup A_2)) + \mu(B\cap(A_1\cup A_2)^c)$$

$$= \mu\big((B\cap A_1)\cup(B\cap A_2\setminus A_1)\big) + \mu(B\cap A_1^c\cap A_2^c)$$

$$\le \mu(B\cap A_1) + \boxed{\mu(B\cap A_1^c\cap A_2) + \mu(B\cap A_1^c\cap A_2^c)} \le \mu(B\cap A_1^c)$$

$$\le \mu(B\cap A_1) + \mu(B\cap A_1^c) \le \mu(B)$$

For finite intersections, let A_1, A_2 be measurable.

Then $A_1\cap A_2 = (A_1^c \cup A_2^c)^c$, A_1^c and A_2^c are μ-meas by previous thm. $A_1^c\cup A_2^c$ is μ-meas by above and $(A_1^c\cup A_2^c)^c$ is μ-meas by the previous thm.

Let A_1 and A_2 be disjoint. Let's show that $\mu(A_1 \cup A_2) = \mu(A_1) + \mu(A_2)$.

$$\mu(A_1 \cup A_2) = \mu\left([A_1 \cup A_2] \cap A_2^c \cup [A_1 \cup A_2] \cap A_2\right)$$
$$= \mu([A_1 \cup A_2] \cap A_2^c) + \mu([A_1 \cup A_2] \cap A_2)$$
$$= \mu(A_1) + \mu(A_2)$$

$$\mu(B \cap (A_1 \cup A_2)) = \mu([(B \cap A_1) \cup (B \cap A_2)] \cap A_2) + \mu([(B \cap A_1) \cup (B \cap A_2)] \cap A_2^c]$$
$$= \mu(B \cap A_2) + \mu(B \cap A_1)$$

Now let's consider infinite unions:

We want to show

$$\mu\left(B \cap \left(\bigcup_{k=1}^{\infty} A_k\right)\right) + \mu\left(B \cap \left(\bigcup_{k=1}^{\infty} A_k\right)^c\right) \leq \mu(B).$$

Note that

$$\mu\left(B \cap \left(\bigcup_{k=1}^{N} A_k\right)\right) + \mu\left(B \cap \left(\bigcup_{k=1}^{\infty} A_k\right)^c\right)$$
$$\leq \mu\left(B \cap \left(\bigcup_{k=1}^{N} A_k\right)\right) + \mu\left(B \cap \left(\bigcap_{k=1}^{N} A_k^c\right)\right)$$
$$\leq \mu(B) \quad \text{by measurability of } \bigcup_{k=1}^{N} A_k$$

Furthermore, observe that $\mu\left(B \cap \left(\bigcup_{k=1}^{N} A_k\right)\right)$
$$= \sum_{k=1}^{N} \mu\left(\left(A_k \setminus \left(\bigcup_{j=1}^{k-1} A_j\right)\right) \cap B\right)$$

$\Rightarrow$

$$\sum^{N} \mu\left(\left[A_k \setminus \left(\bigcup_{j=1}^{k-1} A_j\right)\right] \cap B\right) + \mu\left(B \cap \left(\bigcup_{k=1}^{\infty} A_k\right)^c\right) \leq \mu(B)$$

for all N

$$\Rightarrow \quad \sum^{\infty} \mu\left(\left[A_k \setminus \left(\bigcup_{j=1}^{k-1} A_j\right)\right] \cap B\right) + \mu\left(B \cap \left(\bigcup_{k=1}^{\infty} A_k\right)^c\right) \leq \mu(B).$$

$$\Rightarrow \quad \sum_{k=1}^{\infty} \mu\left(\left[A_k \setminus \left(\bigcup_{j=1}^{k-1} A_j \right)\right] \cap B \right) + \mu\left(B \cap \left(\bigcup_{k=1}^{\infty} A_k \right)^c \right) \le \mu(B).$$

since $\mu\left(\bigcup_{k=1}^{\infty} A_k \right) \le \sum_{k=1}^{\infty} \mu\left(\left[A_k \setminus \left(\bigcup_{j=1}^{k-1} A_j \right)\right] \cap B \right)$ we are done.

The remaining is done similarly.

σ - Algebras

<u>Definition:</u> A collection of subsets $\mathcal{M} \subset \mathcal{P}(X)$ is a σ-Algebra if

i.) $\phi, X \in \mathcal{M}$

ii.) $A \in \mathcal{M} \Rightarrow A^c \in \mathcal{M}$ (closed under complementation)

iii.) $\{A_k\}_{k=1}^{\infty} \subset \mathcal{M} \Rightarrow \bigcup_{k=1}^{\infty} A_k \in \mathcal{M}$. (closed under countable unions).

<u>Note!</u>: If $\mathcal{M}$ is a σ-algebra, and $\{B_k\}_{k=1}^{\infty} \subset \mathcal{M}$, then $\bigcap_{k=1}^{\infty} B_k \in \mathcal{M}$

<u>Thm:</u> If μ is an outer measure on a nonempty set, X, then the collection of all μ-measurable subsets of X is a σ-algebra.

<u>Pf:</u> Already proven.