Recall measurability and the Lebesgue outer measure.

Q1: Is the middle-third Cantor set measurable?

Q2: Are the rationals measurable?

In the case that $\mu(X) < \infty$, are there sets that do not satisfy $\mu(X) - \mu(A^c) = \mu(A)$?

Yes, these are non-measurable sets

Ex: Consider $(\mathbb{T}, |\cdot - \cdot|)$ where $\mathbb{T} := \mathbb{R}/\mathbb{Z} = \{$set of equivalence classes mod 1$\}$.

Consider another equivalence; $\sim$, where

$$x \sim y \iff x - y \equiv q \in \mathbb{Q}.$$

and

$$\mathbb{T}/\sim \; = \{ [x] \mid [x] = \{ y \mid x - y \equiv q \in \mathbb{Q} \} \}.$$

Claim: each equivalence class, $[x]$, contains only countably many elements

This implies $\mathbb{T}/\sim$ is uncountable.

For each equivalence class, choose one representative and define

$$P_0 := \{ \text{set of representatives} \}$$

Now enumerate, $\mathbb{Q} \cap (0,1) = \{q_j\}_{j=1}^{\infty}$, and define

$$P_j := P_0 + q_j \mod 1.$$

Claim: $P_j \cap P_k$ when $j \neq k$.

Now consider an outer measure, μ. suppose that

① $0 < \mu(\mathbb{T}) < \infty$

(2) μ is translation invariant, i.e. $\mu(E) = \mu(E+x)$.

Then
$$\mu(\Pi) = \mu\left(\bigcup_{j=0}^{\infty} P_j\right) \le \sum_{j=0}^{N} \mu(P_j) + \mu\left(\sum_{j=N+1}^{\infty} P_j\right).$$
$$= N\mu(P_0) + \mu\left(\sum_{j=N+1}^{\infty} P_j\right).$$
$$\overset{?}{\le} \mu(\Pi).$$

$$\mu(\Pi) \le N\mu(P_0) \overset{?}{\le} \mu(\Pi). \quad \text{for all } N$$

$\Longleftrightarrow$ $\quad\Rightarrow \mu(\Pi)=0$ or $\mu(\Pi)=\infty$.

We now need to establish some basic properties of measures and measurability.

Thm: Let μ be an outer measure on X.

i.) If $A \subset B \subset X$ then $\mu(A) \le \mu(B)$ (monotonicity)

ii.) A is μ-measurable $\Leftrightarrow$ $X \setminus A$ is μ-measurable.

iii.) ϕ and X are μ-measurable.

If $\mu(A)=0$ then A is μ-measurable

iv.) If $E \subset X$, then each μ-measurable set is also $\mu|_E$ - measurable.