Measure Theory

Outer measures (Exterior measures).

Let $X \neq \phi$. Then we denote the power set of X by

$$\mathcal{P}(X) = \text{the collection of all subsets of } X$$

Def: A mapping $\mu : \mathcal{P}(X) \to [0,\infty]$ is called an outer measure on X provided that

i.) $\mu(\phi) = 0$

ii.) subadditivity: if $A \subset \bigcup_{i=1}^{\infty} A_i$, then

$$\mu(A) \leq \sum_{i=1}^{\infty} \mu(A_i)$$

Examples

① Let $X = \{x_k\}_{k=1}^{\infty}$

and let
$$\mu(E) = \begin{cases} \text{cardinality of } E, & E \neq \phi \\ 0, & E = \phi \end{cases}$$

② Consider the metric space $(\mathbb{R}, |\cdot|)$.

For an interval, $I \subset \mathbb{R}$, let $|I|$ denote the length of I. Define the function

$\mu : \mathcal{P}(\mathbb{R}) \to [0,\infty]$ by

$$\mu(E) = \inf\left\{ \sum_{j=1}^{\infty} |I_j| \;\middle|\; E \subset \bigcup_{j=1}^{\infty} I_j,\ I_j \text{ is an interval} \right\}.$$

Then μ is an outer measure.

<u>Note:</u> If μ is an outer measure on X and $E \subset X$, then μ <u>restricted to E</u> is an outer measure defined by

$$\mu\big|_E(A) := \mu(A \cap E) \qquad \forall A \subset X.$$

<u>Def:</u> A set $A \subset X$ is <u>μ-measurable</u>, if for each set $B \subset X$,

$$\mu(B) = \mu(B \cap A) + \mu(B \setminus A)$$
$$= \mu(B \cap A) + \mu(B \cap A^c).$$

<u>Note:</u> $B = (B \cap A) \cup (B \setminus A)$ so by definition of outer measure $\mu(B) \leq \mu(B \cap A) + \mu(B \setminus A)$.

What is this definition trying to convey?

As Folland notes, the equation

$$\mu(B) = \mu(B \cap A) + \mu(B \setminus A) \quad \text{is equivalent}$$

to

$$\mu(B) - \mu(B \setminus A) = \mu(B \cap A)$$

So in some sense, we are saying that

we can measure A directly or by measuring its complement and subtracting from the measure of the entire space. So why don't we just take

$$\mu(X) - \mu(A^c) = \mu(A) ?$$

$\mu(X)$ can be infinite. In which case, we learn nothing.