Def: (Finite Intersection Property)

A collection $\mathcal{F}$ of sets in (X, ρ) is said to have the <u>finite intersection property</u> if any finite subcollection of $\mathcal{F}$ has non-empty intersection

There is a HW problem on finite intersection property.

Def: • $E \subset X$ is <u>sequentially compact</u> if every sequence $\{x_n\} \subset E$ has a convergent subsequence

• $E \subset X$ is <u>bounded</u> if $\exists x_0 \in X$ and $r > 0$ s.t.

$$B(x_0, r) \supset E$$

• $E \subset X$ is <u>totally bounded</u> if for all $\varepsilon > 0$, $\exists$ a finite collection $\{x_1, \dots, x_n\} \subset X$ s.t.

$$E \subset \bigcup_{i=1}^{n} B(x_i, \varepsilon).$$

Ex: (X, discrete metric) Is not compact. As long as the cardinality of X is infinity.

Ex: $\overline{B(0,1)} \subset \ell^2(\mathbb{N})$. is bounded and closed but not compact or seq. compact

Ex: $\overline{B(0,1)} \subset \ell^\infty(\mathbb{N})$. is bounded and closed but not compact or seq. compact

<u>Thm:</u> Let (X, ρ) be a metric space, $E \subset X$.

The following are equivalent:

i.) E complete and totally bounded.

ii.) E is sequentially compact

iii.) E is compact.

$(i) \Leftrightarrow (ii)$

$(i) + (ii) \Rightarrow (iii)$

$[\text{not } (ii) \Rightarrow \text{not } (iii).] \Rightarrow (iii) \Rightarrow (ii).$