Recall:

Thm: $f \in BV([a,b]) \Rightarrow f$ diff m-a.e.

Recall: From f we construct a measure μ_f on $[a,b]$.

and $\lim_{r \to 0} \frac{\mu_f([x-r, x+r])}{2r} = f'(x)$ m-a.e.

However $d\mu_f \neq f'\,dm$. (i.e. $\mu_f(E) \neq \int_E f'\,dm$)

Ex: Let $f = \begin{cases} 0 & x \in [0, 1/2] \\ 1 & x \in (1/2, 1] \end{cases}$. Then $f \in BV([0,1])$.

$\lim_{r \to 0} \frac{\mu_f([x-r, x+r])}{2r} = 0$ for all $x \neq 1/2$.

However, $\mu_f([0,1]) = f(1) - f(0) = 1 \neq 0 = \int_0^1 f'\,dm$.

Why? Recall that μ_f has a LRN decomposition w.r.t m

$\mu_f = \lambda + \int g\,dm$ and $\frac{\mu_f([x-r, x+r])}{2r} \to g(x)$ m-a.e.

So we've avoided λ when differentiating f.
Unfortunately, when attempting a fundamental theorem of calculus, we must confront the possibility that $\mu_f \not\ll m$.

In the previous example, $\mu_f = \delta_{\{1/2\}}$. So $\mu_f \perp m$.

Thm: If $\mu_f \ll m$ iff f is diff'ble m-a.e.

and $f(b) - f(a) = \int_a^b f'\,dm$.

Def: $f : [a,b] \to \mathbb{R}$ is absolutely continuous if
$\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. if $\{(a_j, b_j)\}_{j=1}^n$ are disjoint intervals.

n $\quad$ n

$\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. if $\{(a_j, b_j)\}_{j=1}^n$ are disjoint intervals.

$$\sum_{j=1}^n |b_j - a_j| < \delta \Rightarrow \sum_{j=1}^n |f(b_j) - f(a_j)| < \varepsilon.$$

Prop: f is absolutely continuous iff $\mu_f \ll m$.

Higher Dimensions

Consider a function $f : \mathbb{R}^d \to \mathbb{R}$, is there a condition similar to BV that guarantees that ∇f exists m.a.e.?

We first note that the graph of a function of bounded variation has finite $\mathcal{H}^1$ measure.

Cor: Let $f : [a,b] \to \mathbb{R}$ be a function and let $G_f := \{ (x, f(x)) \mid x \in [a,b] \}$.

f is differentiable m-a.e. iff the length of G_f is finite.

pf: Parametrize G_f by $z(t) = (x(t), y(t))$ where $x(t) = t$ and $y(t) = f(t)$.

Then G_f has finite length $\Leftrightarrow$ x and y are of bounded variation

x is trivially of bounded variation $\square$

Q: Let $f : [0,1]^2 \to \mathbb{R}$ be continuous, Q: Let $f : [0,1]^2 \to \mathbb{R}$ be continuous, and consider $G_f = \{ (x_1, x_2, f(x_1,x_2)) \mid (x_1,x_2) \in [0,1]^2 \}$.

Is it true that

f is differentiable m-a.e. iff $\mathcal{H}^2(G_f) < \infty$.

A: No, One can create a lot of non differentiability without adding surface area.

However, recall that Lipschitz functions are of bounded variation

Corollary: $f : [a,b] \to \mathbb{R}$ Lipschitz $\Rightarrow$ f is differentiable m-a.e.

(In fact, f Lip $\Rightarrow$ f absolutely continuous).

Moreover, Lipschitz functions have a simple generalization to higher dimensions.

Thm (Rademacher)

If $f : [0,1]^d \to \mathbb{R}$ is Lipschitz, then f is differentiable m-a.e.

Pf: Let $v \in S^{d-1}$ and $z_0 \in [0,1]^d$. Consider the line $\ell_{v,z_0} = \{ y \in [0,1]^d \mid y = z_0 + tv \text{ for some } t \in \mathbb{R} \}$.

The function $g(t) = f(z_0 + tv)$ is Lipshitz on some interval of $\mathbb{R}$.

$\Rightarrow \quad g'$ exists $\mathcal{L}^1$-a.e.

$\Rightarrow \quad \lim_{h \to 0} \frac{f(x+hv) - f(x)}{h}$ exists for $\mathcal{L}^1$-a.e. $x \in l_{v,z_0}$.

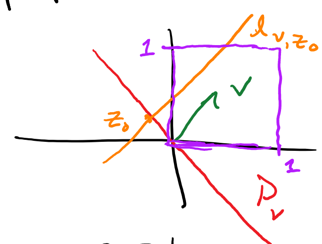
Denote $f_v(x) := \lim_{h\to 0} \frac{f(x+hv) - f(x)}{h}$ the derivative of f in the direction v.

and $N_v = \{ x \in [0,1]^d \mid f_v(x) \text{ does not exist} \}$.

By Fubini:

$$\mathcal{L}^d(N_v) = \int_{P_v} \mathcal{L}^1(N_v \cap l_{v,z_0}) \, d\mathcal{L}^{d-1}(z_0) = \int_{P_v} 0 \, d\mathcal{L}^{d-1}(z_0) = 0.$$

where P_v = plane through 0 perpendicular to v



$\Rightarrow \quad f_v(x)$ exists for $\mathcal{L}^d$-a.e. $x \in [0,1]^d$.

Let $\{e_1, \dots, e_d\}$ be the standard orthonormal basis of $\mathbb{R}^d$.

$\Rightarrow \quad f_{e_i}(x)$ exists for $\mathcal{L}^d$-a.e. $x \in [0,1]^d$

$\Rightarrow \quad \nabla f(x) = (f_{e_1}(x), f_{e_2}(x), \dots, f_{e_d}(x))$ exists for $\mathcal{L}^d$-a.e. $x \in [0,1]^d$

Let $\{v_k\}_{k=1}^\infty \subset S^{d-1}$ be a countable, dense subset of the unit sphere. Then, in fact, since $\mathcal{L}^d(\bigcup N_{v_k}) = 0$, at almost every $x \in [0,1]^d$, $f_{v_k}(x)$ exists for all k.

For any $v \in S^{d-1}$, let $v_{k_j} \xrightarrow{j\to\infty} v \Rightarrow \{v_{k_j}\}$ is Cauchy

and thus

$$|f_{v_{k_j}}(x) - f_{v_{k_m}}(x)| = \left| \lim_{h\to 0} \frac{f(x+hv_{k_j}) - f(x)}{h} - \lim_{h \to 0} \frac{f(x+hv_{k_m}) - f(x)}{h} \right|$$

$$\leq \lim \frac{|f(x+hv_{k_j}) - f(x+hv_{k_m})|}{} \leq \lim \frac{|h| \, \| v_{k_j} - v_{k_m} \|}{}$$

$$\leq \lim_{h\to 0} \frac{|f(x+hv_{k_j}) - f(x+hv_{k_m})|}{|h|} \leq \lim_{h\to 0} \frac{|h|\,\|v_{k_j}-v_{k_m}\|}{|h|}$$

$$\leq \|v_{k_j}-v_{k_m}\|$$

$\Rightarrow \{f_{v_{k_j}}\}$ is Cauchy

$\Rightarrow f_v(x) = \lim_{j\to\infty} f_{v_{k_j}}(x)$ exists.

Therefore, for $\mathcal{L}^d$-almost every $x \in [0,1]^d$,

∇f exists and $f_v(x)$ exists for all $v \in S^{d-1}$

Final step: we need to show that $f_v(x) = v \cdot \nabla f(x)$ for $\mathcal{L}^d$-almost every $x \in [0,1]^d$.

Lemma: Let f be a measurable function.
$f = 0$ m-a.e. iff $\int_{\mathbb{R}^d} f \varphi \, dm(x) = 0$ for all $\varphi \in C_c^\infty(\mathbb{R}^d)$.

The Lemma implies that it suffices to show that

$$\int_{\mathbb{R}^d} f_v \varphi = \int (v \cdot \nabla F)\varphi \quad \text{for all } \varphi \in C_c^\infty(\mathbb{R}^d).$$

$$\int_{\mathbb{R}^d} f_v(x)\varphi(x)\,dx = \int \lim_{h\to 0} \frac{f(x+hv)-f(x)}{h}\varphi(x)\,dx$$

$$= \lim_{h\to 0} \int \frac{f(x+hv)}{h}\varphi(x) - \frac{f(x)}{h}\varphi(x)\,dx$$

Dominated Convergence Theorem

$$= \lim_{h\to 0} \int \frac{1}{h} f(x)\varphi(x-hv) - \int \frac{1}{h} f(x)\varphi(x)\,dx$$

$$= \lim_{h\to 0} \int \frac{1}{h}\left(\varphi(x-hv) - \varphi(x)\right) f(x)\,dx$$

$$= \int \lim_{h\to 0} \frac{(\varphi(x-hv) - \varphi(x))}{h} f(x)\,dx$$

$$= -\int \varphi_v(x)\, f(x)\, dx$$

$$= -\int \left(v \cdot \nabla \varphi(x)\right) f(x)\, dx$$

$$= -\sum_{i=1}^{d} v_i \int \varphi_{e_i}(x) f(x)\, dx$$

$$= \sum_{i=1}^{d} v_i \int \varphi(x)\, f_{e_i}(x)\, dx = \int \varphi(x) \left(v \cdot \nabla f(x)\right) dx \qquad \Box.$$