Recall:

$$f \in BV([a,b]) \iff \sup_{\{t_i\}_{i=0}^n} \sum_{i=1}^n |f(t_i) - f(t_{i-1})| < \infty.$$

Thm: $f \in BV([a,b]) \Rightarrow f = g - h$ where g and h are monotonic.

Thm: Let $f : \mathbb{R} \to \mathbb{R}$ be a monotonically increasing function. Then

a.) f is discontinuous at at most countably many points

b.) f is differentiable m-a.e.

---

pf: (a) Done in HW 1.

(b) Prove Two ways.

First Way

Define $g(x) = \lim_{y \to x^+} f(y)$.

The g is right-continuous. Define

$$\mu_g(E) := \inf\left\{ \sum_{j=1}^{\infty} g(b_j) - g(a_j) \,\middle|\, E \subset \bigcup_{j=1}^{\infty} (a_j, b_j] \right\}$$

(Right-continuity necessary for countable additivity)

Claim: μ_g is Radon

Therefore, the differentiation theorem implies $\exists F \in L^1_{loc}$ s.t.

$$\frac{g(x+h) - g(x)}{h} = \frac{\mu_g((x, x+h])}{m((x, x+h])} \xrightarrow{h \to 0^+} F(x) \quad m\text{-a.e.}$$

$\rightarrow g' = F$

It suffices to show that if $G = g - F$, then $G' = 0$ m-a.e.

Note that $G \geq 0$ and $G \neq 0$ on the set, D, constructed in part (a). Enumerate $D = \{x_j\}_{j=1}^{\infty}$

Consider the measure

$$\mu := \sum_{j=1}^{\infty} G(x_j)\,\delta_{x_j}$$

<u>Claim:</u> μ is Radon. and $\mu \perp m$

This implies $\frac{\mu([x-h, x+h])}{h} \xrightarrow{h \to 0^+} 0$ m-a.e.

$$\Rightarrow \left| \frac{G(x+h) - G(x)}{h} \right| \le \frac{\mu([x-h, x+h])}{|h|} \to 0 \quad m\text{-a.e.}$$

<u>Second Way</u>

Define the Dini numbers for a function $f: \mathbb{R} \to \mathbb{R}$

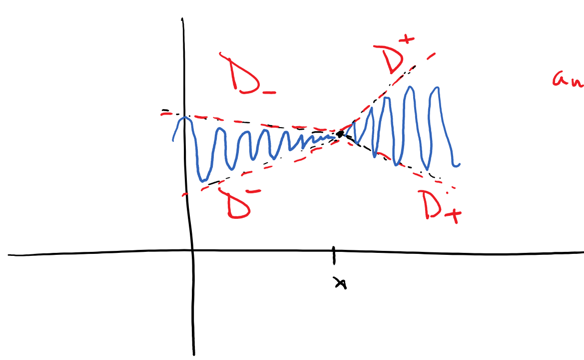
$$D^+ f(x) = \limsup_{h \to 0^+} \frac{f(x+h) - f(x)}{h}$$

$$D_+ f(x) = \liminf_{h \to 0^+} \frac{f(x+h) - f(x)}{h}$$

$$D^- f(x) = \limsup_{h \to 0^-} \frac{f(x+h) - f(x)}{h}$$

$$D_- f(x) = \liminf_{h \to 0^-} \frac{f(x+h) - f(x)}{h}.$$

<u>Picture</u>



$D^+ \ge D_+$ and $D^- \ge D_-$

It suffices to prove that i) $D^+ f(x) < \infty$ for m-a.e. x and

ii) $D^+ f(x) \le D_- f(x)$ for m-a.e. x

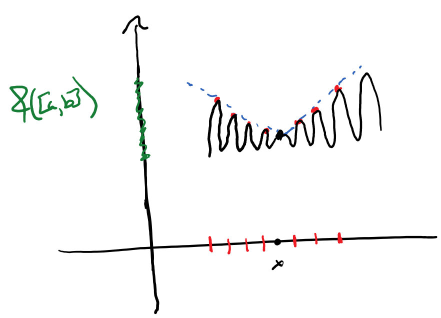
ii) is sufficient because $g(x) = -f(-x)$ is monotonically increasing and $D^+g(x) \le D_-g(x) \iff D^-f(x) \le D_+f(x)$

$$\Rightarrow \quad D^-f(x) \le D_+f(x) \le D^+f(x) \le D_-f(x) \le D^-f(x)$$

$$\Rightarrow \quad D^-f(x) = D^+f(x) = D_+f(x) = D_-f(x) \quad \Rightarrow \quad f' \text{ exists.}$$

or $f' = \infty$

Proof of (ii) by picture



Cover $f([a,b])$ by disjoint intervals

$$D^+f \cdot \left(\sum (b_j - a_j)\right) \le \sum f(b_j) - f(a_j) \le D_-f(x) \cdot \left(\sum (b_j - a_j)\right)$$

$$\Rightarrow \quad D^+f \le D_-f(x).$$

To prove (i), we show that $\int_a^b f' \, dm \le f(b) - f(a)$

which would imply $m(\{f' = \infty\}) = 0$

Observe

$$\int_a^b f' \, dm = \int_a^b \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \, dm(x)$$

Suppose f is right continuous at b. Then

$$\int_a^b \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \, dm(x) = \int_a^b \lim_{h \to 0^+} \frac{f(x+h) - f(x)}{h} \, dm(x)$$

$$\overset{\text{Fatou}}{\le} \liminf_{h \to 0^+} \int_a^b \frac{f(x+h) - f(x)}{h} \, dm(x)$$

$$= \liminf_{h \to 0^+} \frac{1}{h} \int_a^b f(x+h) \, dm(x) - \frac{1}{h} \int_a^b f(x) \, dm(x)$$

$$= \liminf_{h \to 0^+} \frac{1}{h} \int_{a+h}^{b+h} f(x) \, dm(x) - \frac{1}{h} \int_a^b f(x) \, dm(x)$$

$$= \liminf_{h \to 0^+} \frac{1}{h} \int_b^{b+h} f(x) \, dm(x) - \frac{1}{h} \int_a^{a+h} f(x) \, dm(x)$$

Since f is right continuous at b, $\liminf_{h \to 0^+} \frac{1}{h} \int_b^{b+h} f(x) \, dm(x) = f(b)$

and since f is monotonically increasing, $\frac{1}{h}\int_a^{a+h} f(x)\,dm(x) \geq \frac{1}{h}\int_a^{a+h} f(a)\,dm(x) = f(a)$

$$\Rightarrow \quad \liminf_{h\to 0^+} \frac{1}{h}\int_a^{b+h} f(x)\,dm(x) - \frac{1}{h}\int_a^{a+h} f(x)\,dm(x) \leq f(b) - f(a).$$

Similarly for f left continuous at a.

If f is left continuous at b and right continuous at a then consider sequences of points of continuity of f, $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ then

$$\lim_{n\to\infty} \int_{a_n}^{b_n} f' \leq \lim_{n\to\infty} f(b_n) - f(a_n) \to f(b) - f(a)$$

MCT

$$\int_a^b f'$$

$\square$