Def: (Bounded Variation)

Let $f : [a,b] \to \mathbb{R}$. f is said to be of <u>Bounded Variation</u> if

$$\sup \left\{ \sum_{i=1}^{n} |f(t_i) - f(t_{i-1})| \;\middle|\; P = \{t_i\}_{i=0}^{n} \text{ partitions } [a,b] \right\} < \infty.$$

This class of functions is denoted by

$$BV([a,b])$$

Ex:
- Monotone, bounded functions
- $C^1([a,b]) = \{ f : [a,b] \to \mathbb{R} \mid f \text{ is differentiable and } f' \in C([a,b]) \}$
- Lipschitz functions.

Thm: A curve, $\gamma \subset \mathbb{R}^2$, parametrized by $z(t) = (x(t), y(t))$, $t \in [a,b]$ has finite length iff $x, y \in BV([a,b])$.

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pf: γ having finite length is defined by

$$\sup \left\{ \sum_{i=1}^{n} \|z(t_{i+1}) - z(t_i)\| \;\middle|\; P = \{t_i\}_{i=0}^{n} \text{ is a partition of } [a,b] \right\}.$$

Now $\frac{1}{2} |x(t_{i+1}) + x(t_i)| + |y(t_{i+1}) - y(t_i)| \le \|z(t_{i+1}) - z(t_i)\| \le |x(t_{i+1}) - x(t_i)| + |y(t_{i+1}) - y(t_i)|$

$\square$.

Variations

Def: Let $f : [a,b] \to \mathbb{R}$, $x \in [a,b]$ define

- Total Variation of f on $[a,x]$

$$T_f(a,x) := \sup \left\{ \sum_{i=1}^{n} |f(t_i) - f(t_{i-1})| \;\middle|\; P = \{t_i\}_{i=1}^{n} \text{ is a partition of } [a,x] \right\}.$$

• Positive Variation of f on $[a,x]$

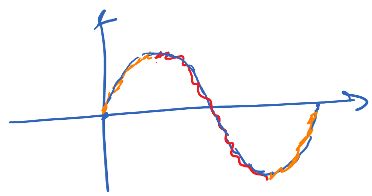
$$P_f(a,x) := \sup\left\{ \sum_{i=1}^{n} \left(f(t_i) - f(t_{i-1})\right)^+ \;\middle|\; P = \{t_i\}_{i=1}^{n} \text{ is a partition of } [a,x] \right\}$$

• Negative Variation of f on $[a,x]$

$$N_f(a,x) := \sup\left\{ \sum_{i=1}^{n} \left(f(t_i) - f(t_{i-1})\right)^- \;\middle|\; P = \{t_i\}_{i=1}^{n} \text{ is a partition of } [a,x] \right\}.$$

Recall: $a^+ = \max(a, 0)$ $\qquad a^- = \max(-a, 0)$

Ex: Let $f(x) = \sin(x)$ on $[0, 2\pi]$.



$$P_f(0,x) = \left(f \cdot \chi_{[0,\pi/2]} + \left(f - f(\tfrac{3\pi}{2})\right)\chi_{(3\pi/2,\, 2\pi]}\right)\chi_{[0,x]}.$$

$$N_f(0,x) = -\left(\left(f - f(\pi/2)\right)\chi_{(\pi/2,\, 3\pi/2]}\right)\chi_{[0,x]}$$

$$P_f(\tfrac{\pi}{2}, x) = \left(f \cdot \chi_{[0,\pi/2)} + \left(f - f(\tfrac{3\pi}{2})\right)\chi_{(3\pi/2,\, 2\pi]}\right)\chi_{[\pi/2,\, x]}.$$

$$N_f(\tfrac{\pi}{2}, x) = -\left(\left(f - f(\tfrac{\pi}{2})\right)\chi_{[\pi/2,\, 3\pi/2)}\right)\chi_{[\pi/2,\, x]}.$$

Proposition Suppose $T_f(a,b) < \infty$, if $x \in [a,b]$, then

i) $f(x) - f(a) = P_f(a,x) - N_f(a,x)$

ii.) $T_f(a,x) = P_f(a,x) + N_f(a,x)$

iii.) P_f and N_f are increasing functions.

Pf: i.) $\exists\, P_j = \{t_i^j\}$ s.t. $\sum \left(f(t_i^j) - f(t_{i-1}^j)\right)^+ \xrightarrow{j\to\infty} P_f(a,x)$

and $\sum \left(f(t_i^j) - f(t_{i-1}^j)\right)^- \xrightarrow{j\to\infty} N_f(a,x).$

Moreover, $f(x) - f(a) = \sum \left(f(t_i^j) - f(t_{i-1}^j)\right)^+ - \sum \left(f(t_i^j) - f(t_{i-1}^j)\right)^-$

for all j

ii) By common refinements.

iii) Let $x < y$

for any partition of $[a,x]$.

$$\sum_{i=0}^{n} (f(t_i) - f(t_{i-1}))^+ \leq \sum_{i=0}^{n} (f(t_i) - f(t_{i-1}))^+ + (f(y) - f(t_n))^+$$

$\Rightarrow \quad P_f(a,x) \leq P_f(a,y).$

Similar is true for N_f. $\square$

Thm: $f \in BV([a,b])$ iff. f is the difference of 2 monotone, bounded, real-valued functions on $[a,b]$.

Pf: $(\Rightarrow)$ $f \in BV([a,b]) \Rightarrow T_f(a,b) < \infty$

$\Rightarrow$ by proposition that

$f(x) - f(a) = P_f(a,x) - N_f(a,x)$

$\Rightarrow f(x) = (P_f(a,x) + f(a)) - N_f(a,x).$

$(\Leftarrow)$. $f = g - h$

$$\Rightarrow \sum |f(t_i) - f(t_{i-1})| \leq \sum |g(t_i) - g(t_{i-1})| + \sum |h(t_i) - h(t_{i-1})|$$

$$\leq g(b) - g(a) + (h(a) - h(b)).$$

$\square$.