The Lebesgue Differentiation Theorem

Thm: If $f \in L^1(\mathbb{R}^d)$ then

$$(*) \quad \lim_{r \to 0} \frac{1}{m(B(x,r))} \int_{B(x,r)} f \, dm = f(x) \quad \text{for } m\text{-a.e. } x$$

In fact,

$$(**) \quad \lim_{r \to 0} \frac{1}{m(B(x,r))} \int_{B(x,r)} |f(y) - f(x)| \, dm(y) = 0 \quad \text{for } m\text{-a.e. } x.$$

Observation: f need only be locally integrable

Consider the following class of functions

$$L^1_{loc}(\mathbb{R}^d) = \left\{ f : \mathbb{R}^d \to \mathbb{R} \;\middle|\; \text{for every bounded, measurable } K \subset \mathbb{R}^d, \ f \chi_K \in L^1(\mathbb{R}^d) \right\}.$$

Corollary: If $f \in L^1_{loc}(\mathbb{R}^d)$ then

$$\lim_{r \to 0} \frac{1}{m(B(x,r))} \int_{B(x,r)} |f(y) - f(x)| \, dm(y) = 0 \quad \text{for } m\text{-a.e. } x.$$

Pf: Let $B_k := B(0,k)$. and $f_k := f \cdot \chi_{B_k}$.

Then the Lebesgue differentiation theorem holds for f_k and $f_k \to f$. $\square$

The Differentiation theorem for general measures

Def: (Radon Measures).

We say that a Borel measure, μ, on a metric space (X,ρ) is Radon if

i.) $\mu(K) < \infty$ for all compact sets, K.

ii.) $\mu(U) = \sup\{\mu(K) \mid K \subset U,\ K \text{ compact}\}$ for U open

iii.) $\mu(E) = \inf\{\mu(U) \mid E \subset U,\ U \text{ open}\}$ for E measurable

Ex:

- Lebesgue measure
- Hausdorff measure restricted to s-dimensional set
- If $f: \mathbb{R} \to \mathbb{R}$ is continuous and increasing

$$\mu_f(E) := \inf\left\{ \sum_{j=1}^{\infty} f(b_j) - f(a_j) \;\middle|\; E \subset \bigcup_{j=1}^{\infty} [a_j, b_j] \right\}.$$

Thm: Let ν be a Radon measure on $\mathbb{R}^d$, and let $d\nu = d\lambda + f\,dm$ be its Lebesgue-Radon-Nikodym representation w.r.t. m. Then for m-a.e. $x \in \mathbb{R}^d$.

$$\lim_{r \to 0} \frac{\nu(B(x,r))}{m(B(x,r))} = f(x).$$

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pf:

Observe that $\displaystyle \lim_{r\to 0} \frac{\nu(B(x,r))}{m(B(x,r))} = \lim_{r \to 0}\left(\frac{\lambda(B(x,r))}{m(B(x,r))} + \frac{\int_{B(x,r)} f\,dm}{m(B(x,r))} \right)$

We already know $\displaystyle \frac{1}{m(B(x,r))} \int_{B(x,r)} f\,dm \to f(x)$ m-a.e.

$\boxed{\text{As long as } f \in L^1_{loc} \text{ which we know since } \nu \text{ is Radon}}$ *

So it suffices to show that

$$\lim_{r \to 0} \frac{\lambda(B(x,r))}{m(B(x,r))} = 0 \quad m\text{-a.e.}$$

By definition of λ, $\lambda \perp m$, so there exists $A \subset \mathbb{R}^d$ s.t. $\lambda(A)=0$ and $m|_A = m$.

Let $\alpha > 0$ and define

$$E_\alpha := A \cap \left\{ x \in \mathbb{R}^d \;\middle|\; \limsup_{r\to 0} \frac{\lambda(B(x,r))}{m(B(x,r))} > \alpha \right\}.$$

(Note: On E_α, m is quantitatively absolutely continous with respect to λ, but since m and λ are mutually singular, E_α must be measure zero)

It suffices to show that $m(E_\alpha) = 0$.

Claim: ν regular $\Rightarrow$ λ and $f\,dm$ are regular.

Let $\varepsilon > 0$, and let $U \supset E_\alpha$ be such that

$$\lambda(U) \leq \lambda(E_\alpha) + \varepsilon \Rightarrow \lambda(U) \leq \varepsilon.$$

For every $x \in E_\alpha$ $\exists \{r_i\}_{i=1}^\infty$ s.t. $r_i \to 0$ and.

$$\frac{\lambda(B(x,r_i))}{m(B(x,r_i))} > \alpha. \quad \text{for all } i.$$

Let $$\mathcal{B} = \bigcup_{x\in E_\alpha} \left\{ B(x,r_i) \;\middle|\; B(x,r_i) \subset U \right\}.$$ (Note: The sequence r_i depends on x).

$\mathcal{B}$ is a fine cover of E_α. Therefore, by Vitali covering theorem, $\exists \{B_k\}_{k=1}^\infty \subset \mathcal{B}$ s.t.

$$m\left(E_\alpha \setminus \left(\bigcup_{k=1}^\infty B_k\right)\right) = 0.$$

$$\Rightarrow \quad m(E_\alpha) \le \sum_{k=1}^{\infty} m(B_k) \le \frac{1}{\alpha}\sum_{k=1}^{\infty} \lambda(B_k) = \frac{1}{\alpha}\lambda\Big(\bigcup_{k=1}^{\infty} B_k\Big)$$

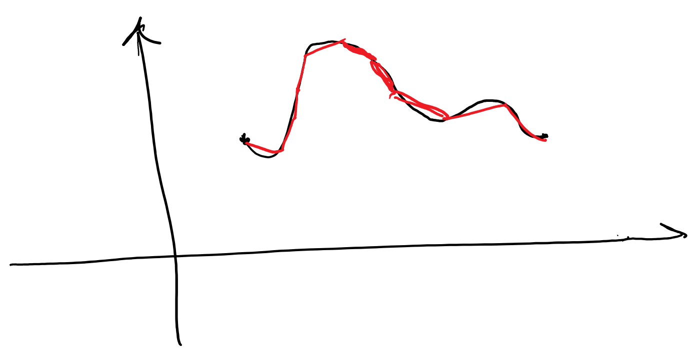
$$\le \frac{1}{\alpha}\lambda(U) \le \frac{\varepsilon}{\alpha}.$$

ε being arbitrary implies $m(E_\alpha) = 0$ $\square$.

Functions of Bounded Variation

Challenge: Can we construct a measure-theoretic notion of a differentiable function?

Consider the graph of a function $f : \mathbb{R} \to \mathbb{R}$



Instead of approximating the graph of a function by a line as we do for differentiable functions, we would like to cumulatively approximate a function by line segments.

Def: (Bounded Variation)

Let $f : [a,b] \to \mathbb{R}$. f is said to be of Bounded Variation

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Let $f:[a,b]\to\mathbb{R}$. f is said to be of Bounded Variation if

$$\sup\left\{ \sum_{i=1}^{n} |f(t_i)-f(t_{i-1})| \;\middle|\; P=\{t_i\}_{i=0}^{n} \text{ partitions } [a,b] \right\} < \infty.$$

This class of functions is denoted by

$$BV([a,b])$$

Ex:
- Monotone, bounded functions
- $C^1([a,b]) = \{ f:[a,b]\to\mathbb{R} \mid f \text{ is differentiable and } f' \in C([a,b]) \}$
- Lipschitz functions.