Def: (Hardy-Littlewood maximal function)

Let $f \in L^1(\mathbb{R}^d)$ the Hardy-Littlewood maximal function (maximal function) of f is defined by

$$(Mf)(x) := \sup \left\{ \frac{1}{m(B)} \int_B |f| \, dm \;\middle|\; \begin{array}{l} x \in B \\ B \text{ is a ball} \end{array} \right\}$$

Thm: Suppose $f \in L^1(\mathbb{R}^d)$ then

i.) Mf is measurable

ii.) $Mf(x) < \infty$ for m-a.e. $x \in \mathbb{R}^d$

iii.) Mf is in "weak L^1", I.e. Mf satisfies the following weak L^1 inequality

$$(*) \quad \forall \alpha > 0 \qquad m(\{x \in \mathbb{R}^d \mid Mf(x) > \alpha\}) \le \frac{5^d}{\alpha} \|f\|_{L^1(\mathbb{R}^d)}.$$

Equivalently $\quad \sup_{\alpha > 0} \alpha \cdot m(\{Mf > \alpha\}) \le 5^d \|f\|_{L^1(\mathbb{R}^d)}.$

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pf: i.) Let $E_\alpha := \{Mf > \alpha\}$

Claim: E_α is open.

Let $x \in E_\alpha \Rightarrow \exists B^{\text{open}}$ s.t. $x \in B$ and

$$\frac{1}{m(B)} \int_B |f| \, dm > \alpha.$$

Since B is open, $\exists B(x,r) \subset B$ and for all $y \in B(x,r)$

$$Mf(y) \ge \frac{1}{m(B)} \int_B |f| \, dm > \alpha.$$

ii.) Follows from (iii) in the following way:

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$$\{Mf = \infty\} = \bigcap_{n=1} \{Mf > n\}$$

$$\Rightarrow \quad m(\{Mf = \infty\}) = \lim_{n\to\infty} m(\{Mf > n\})$$

$$\leq \lim_{n\to\infty} \frac{\|f\|_{L^1}}{n} = 0.$$

iii) Again consider

$$E_\alpha := \{Mf > \alpha\}.$$

For every $x \in E_\alpha$, $\exists B_x$ s.t. $x \in B_x$

$$\frac{1}{m(B_x)} \int_{B_x} |f| \, dm > \alpha.$$

Let $\mathcal{B}_\alpha = \{B_x\}_{x \in E_\alpha}$.

<u>Observe:</u> $\sup\{\operatorname{diam}(B) \mid B \in \mathcal{B}_\alpha\} < \infty$

since $\frac{1}{m(B)} \int_B |f| \, dm > \alpha \Rightarrow m(B) < \frac{\int_B |f| \, dm}{\alpha}$

$$\Rightarrow m(B) < \|f\|_{L^1} / \alpha$$

$$\Rightarrow \sup\{\operatorname{diam}(B) \mid B \in \mathcal{B}_\alpha\} \lesssim \left(\frac{\|f\|_{L^1}}{\alpha}\right)^{1/d}$$

5r Covering Lemma now implies that

$\exists \underset{\text{disjoint}}{\{B(x_i, r_i)\}_{i=1}^\infty} \subset \mathcal{B}_\alpha$ s.t.

$$\bigcup_{B \in \mathcal{B}_\alpha} B \subset \bigcup_{i=1}^\infty B(x_i, 5r_i)$$

$$\Rightarrow m(E_\alpha) \leq \sum_{i=1}^\infty m(B(x_i, 5r_i)) \leq 5^d \sum_{i=1}^\infty m(B(x_i, r_i))$$

$$< 5^d \sum_{i=1}^\infty \frac{1}{\alpha} \int_{B(x_i, r_i)} |f| \, dm$$

$$= 5^d \frac{1}{\alpha} \int_{\bigcup_{i=1}^\infty B(x_i, r_i)} |f| \, dm$$

$$\leq \frac{5^d}{\alpha} \|f\|_{L^1(\mathbb{R}^d)}. \qquad \square.$$

The Lebesgue Differentiation Theorem

Thm: If $f \in L^1(\mathbb{R}^d)$ then

$$(*) \quad \lim_{r \to 0} \frac{1}{m(B(x,r))} \int_{B(x,r)} f \, dm = f(x) \quad \text{for } m\text{-a.e. } x$$

In fact,

$$(**) \quad \lim_{r \to 0} \frac{1}{m(B(x,r))} \int_{B(x,r)} |f(y) - f(x)| \, dm(y) = 0 \quad \text{for } m\text{-a.e. } x.$$

Notes:

- The points in $\mathbb{R}^d$ at which $(**)$ holds are called the Lebesgue points of f.
- If f is continuous with compact support then it is clear that $(*)$ and $(**)$ hold for all x.
- $$\frac{1}{m(B(x,r))} \int_{B(x,r)} f \, dm \leq Mf(x) \quad \text{for all } x \in \mathbb{R}^d$$

  This also suggests that we can generalize $(*)$ to the following condition

  $$\lim_{\substack{m(B) \to 0 \\ x \in B \\ B \text{ a ball}}} \int_B f \, dm = f(x) \quad m\text{-a.e.}$$

  or to different types of shrinking sets as in Folland:

  $$\lim_{\substack{m(Q) \to 0 \\ x \in Q \\ Q \text{ a cube}}} \int_Q f \, dm = f(x) \quad m\text{-a.e.}$$

Pf: For $\alpha > 0$

Let $$E_\alpha = \left\{ x \in \mathbb{R}^d \;\middle|\; \limsup_{r \to 0} \frac{1}{m(B(x,r))} \int_{B(x,r)} |f(x) - f(y)| \, dm(y) > 2\alpha \right\}.$$

Goal: Show $m(E_\alpha)=0$ for all $\alpha>0$.

Claim: $\forall \varepsilon>0$, $\exists g \in C_c(\mathbb{R}^d)$ $\left(C_c(\mathbb{R}^d) = \left\{ g: \mathbb{R}^d \to \mathbb{R} \;\middle|\; \begin{array}{l} g \text{ is continuous} \\ \text{and compactly} \\ \text{supported} \end{array} \right\} \right)$.

s.t. $\|f-g\|_{L^1(\mathbb{R}^d)} < \varepsilon$.

Assuming the claim holds, let g be such a function for f.

Observe:
$$\frac{1}{m(B(x,r))} \int_{B(x,r)} |f(y)-f(x)| \, dm(y)$$
$$\leq \frac{1}{m(B(x,r))} \int_{B(x,r)} |f(y)-g(y)| \, dm(y) + \frac{1}{m(B(x,r))} \int_{B(x,r)} |g(x)-g(y)| \, dm(y) + \frac{1}{m(B(x,r))} \int_{B(x,r)} |f(x)-g(x)| \, dm(y)$$
$$= \frac{1}{m(B(x,r))} \int_{B(x,r)} |f(y)-g(y)| \, dm(y) + \frac{1}{m(B(x,r))} \int_{B(x,r)} |g(x)-g(y)| \, dm(y) + |f(x)-g(x)|.$$
$$\leq M(|f-g|)(x) + \frac{1}{m(B(x,r))} \int_{B(x,r)} |g(x)-g(y)| \, dm(y) + |f(x)-g(x)|.$$

$$\Rightarrow m(E_\alpha) \leq m(\{ M(|f-g|) > \alpha/3 \}) + m\left(\left\{ \limsup_{r\to 0} \frac{1}{m(B(x,r))} \int_{B(x,r)} |g(x)-g(y)| \, dm(y) > \frac{\alpha}{3} \right\}\right) + m(\{ |f(x)-g(x)| > \alpha/3 \})$$

$$\leq \frac{3}{\alpha} 5^d \|f-g\|_{L^1} + 0 + \frac{3}{\alpha} \|f-g\|_{L^1} < 3(5^d+1) \cdot \frac{\varepsilon}{\alpha}.$$

Since ε is arbitrarily small, we are done. $\square$