Thm: (Vitali Covering Theorem)

Let μ be a positive, Borel measure on $(\mathbb{R}^d, \mathcal{M})$ satisfying

- $\mu(E) = \inf\{\mu(U) \mid U \text{ open}, E \subset U\}$, for $E \in \mathcal{M}$
- $\exists\, C > 1$ s.t.

$$\mu(B(x,5r)) \leq C\mu(B(x,r)) \quad \text{(Doubling Condition)}$$

for all $x \in \mathbb{R}^d$, $r \in (0,\infty)$.

Let $A \in \mathcal{M}$, $\mu(A) < \infty$, and let $\mathcal{B}$ be a family of closed balls satisfying

$$(*) \qquad \inf\{\operatorname{diam}(B) \mid x \in B\} = 0 \quad \text{for all } x \in A.$$

Then there exists a disjoint, countable subcollection

$$\{B_i\}_{i=1}^{\infty} \subset \mathcal{B} \quad \text{such that}$$

$$\mu\left(A \setminus \bigcup_{i=1}^{\infty} B_i\right) = 0$$

(In fact, for all $\varepsilon > 0$, $\exists\, \{B_i\}_{i=1}^{N}$ s.t.

$$\mu\left(A \setminus \bigcup_{i=1}^{N} B_i\right) < \varepsilon \text{).}$$

Examples of measures satisfying the hypotheses:

- Lebesgue measure on $\mathbb{R}^d$
- s-dimensional Hausdorff measure restricted to an s-dimensional set.

Pf: Let $A \in M$, and let $U \supset A$ be an open set satisfying
$$\mu(U) \leq \left(1 + \frac{1}{100c}\right)\mu(A).$$
Consider the subcollection of $\mathcal{B}$, $\mathcal{B}_0$, defined by
$$\mathcal{B}_0 := \{ B \in \mathcal{B} \mid B \subset U \text{ and } \operatorname{diam}(B) < 1 \}$$
(so that we can apply 5r Lemma)

By the 5r-covering Lemma, there exists $\{B(x_i, r_i)\}_{i=1}^{\infty} \subset \mathcal{B}_0$ s.t.

- $B(x_i, r_i) \cap B(x_j, r_j) = \emptyset$
- $A \subset \bigcup_{B \in \mathcal{B}_0} B \subset \bigcup_{i=1}^{\infty} B(x_i, 5r_i)$

$$\Rightarrow \mu(A) \leq \mu\left(\bigcup_{i=1}^{\infty} B(x_i, 5r_i)\right)$$
$$\leq \sum_{i=1}^{\infty} \mu(B(x_i, 5r_i))$$
$$\leq C \sum_{i=1}^{\infty} \mu(B(x_i, r_i))$$

and thus $$\sum_{i=1}^{\infty} \mu(B(x_i, r_i)) \geq \frac{\mu(A)}{C}.$$

$$\Rightarrow \exists N_1 \text{ s.t. } \sum_{i=1}^{N_1} \mu(B(x_i, r_i)) \geq \frac{\mu(A)}{2C}.$$

Let $A_1 := A \setminus \bigcup_{i=1}^{N_1} B(x_i, r_i)$, Note: $A_1 \subset U \setminus \bigcup_{i=1}^{N_1} B(x_i, r_i)$

$$\Rightarrow \mu(A_1) \leq \mu\left(U \setminus \bigcup_{i=1}^{N_1} B(x_i, r_i)\right).$$
$$= \mu(U) - \sum_{i=1}^{N_1} \mu(B(x_i, r_i))$$

$$\le \left(1+\frac{1}{100c}\right)\mu(A) - \frac{1}{2c}\mu(A)$$

$$= \left(1+\frac{1}{100c}-\frac{1}{2c}\right)\mu(A)$$

Let $\eta = 1+\frac{1}{100c}-\frac{1}{2c}$, then $\eta < 1$,

and $\mu(A_1) \le \eta(A)$.

We repeat with A_1 to construct $\{B(x_i,r_i)\}_{i=N_1+1}^{N_2}$

s.t. if $A_2 = A_1 \setminus \bigcup_{i=N_1+1}^{N_2} B(x_i,r_i) \subset \left[U_1 \setminus \bigcup_{i=1}^{N_1} B(x_i,r_i)\right] \setminus \bigcup_{i=N_1+1}^{N_2} B(x_i,r_i)$

$$\mu(A_2) \le \eta\,\mu(A_1) \le \eta^2\mu(A)$$

Inductively we construct

A_k with $\mu(A_k) \le \eta^k \mu(A) \xrightarrow{k\to\infty} 0$

and $\mu\left(A \setminus \bigcup_{i=1}^{\infty} B_i\right) = 0$ $\square$.

We will use an important tool in analysis

to finally prove that $\frac{\nu(B(x,r))}{\mu(B(x,r))} \xrightarrow{r\to 0} \frac{d\nu}{d\mu}$. when $\nu \ll \mu$

This tool is known as the Hardy-Littlewood maximal function.

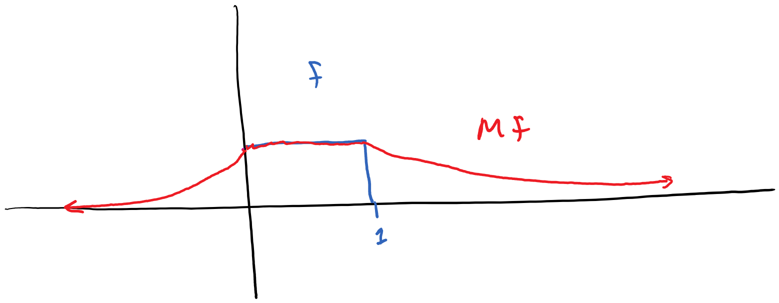
Def: (Hardy-Littlewood maximal function)

Let $f \in L^1(\mathbb{R}^d)$ the Hardy-Littlewood maximal function (maximal function) of f is defined by

$$(Mf)(x) := \sup\left\{ \frac{1}{m(B)}\int_B |f|\,dm \;\middle|\; \begin{array}{l} x \in B \\ B \text{ is a ball} \end{array} \right\}$$

Ex: Let $f = \chi_{[0,1]}$.

$$Mf(x) = \begin{cases} 1 & x \in [0,1]. \\ 1/x & x > 1 \\ \frac{1}{1-x} & x < 0. \end{cases}$$



Note that this example demonstrates that we should <u>not</u> expect $Mf \in L^1$