Q: In what sense is the Radon-Nikodym derivative a "derivative" in classical terms?

Recall: Fundamental theorem of calculus

If $F(x) = \int_0^x f(t)\,dt$ $F'(x) = f(x)$.

So if $\nu \ll \mu$, $\nu(E) = \int_E \frac{d\nu}{d\mu}\,d\mu$

does the following hold:

$$\frac{\nu(B(x,r))}{\mu(B(x,r))} \xrightarrow{r\to 0} \frac{d\nu}{d\mu}(x) \;?$$

In order to confirm this we start with covering lemmas. We will need to restrict our study to Euclidean space.

The covering lemmas will allow us to decompose measurable sets into nicer sets up to exceptional sets of arbitrarily small size.

Covering Lemmas/Theorems

Lemma (5r Covering Lemma)

Let $\mathcal{B}$ be a collection of nondegenerate, closed balls. contained in a bounded set in $\mathbb{R}^d$ satisfying

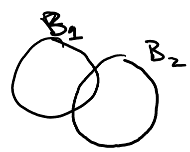
$$\sup\{\operatorname{diam}(B) \mid B \in \mathcal{B}\} < \infty.$$

Then there exists a countable subcollection, $\{B(x_i, r_i)\}_{i=1}^{\infty} \subset \mathcal{B}$, s.t.

- $\bigcup_{B \in \mathcal{B}} B \subset \bigcup_{i=1}^{\infty} B(x_i, 5r_i)$

- $B(x_i, r_i) \cap B(x_j, r_j) = \emptyset$

Ex:
- Why 5?
- Why does one need a scaling factor at all?

  Consider  $\mathcal{B} := \{B_1, B_2\}$.

- Why do we need $\sup\{\operatorname{diam}(B)\} < \infty$?

  Consider $\{B(0,n)\}_{n=1}^{\infty}$.

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pf:

Let $M := \sup\left\{ \frac{\operatorname{diam}(B)}{2} \;\middle|\; B \in \mathcal{B} \right\} < \infty.$

Define $A = \{ x \in \mathbb{R}^d \mid x \text{ is the center of a ball}, B \in \mathcal{B} \}$.

For each $x \in A$, define $r(x) = \sup\{ r \mid B(x,r) \in \mathcal{B} \}$.

Now decompose A in the following way

$$A = \bigcup_{k=1}^{\infty} A_k \quad \text{where} \quad A_k = \left\{ x \in A \;\middle|\; \left(\tfrac{7}{8}\right)^k \cdot M < r(x) \le \left(\tfrac{7}{8}\right)^{k-1} \cdot M \right\}.$$

We proceed by induction

step 1: Consider $A_1 = \{ x \in A \mid \frac{7}{8} \cdot M < r(x) \le M \}$.

Choose $x_1 \in A_1$ and let $r_1 > \frac{7}{8} r(x_1)$ be such that $B(x_1, r_1) \in \mathcal{B}$.

If $A_1 \subset B(x_1, 3r_1)$ proceed to step 2.

If $A_1 \setminus B(x_1, 3r_1) \neq \emptyset$, then pick $x_2 \in A_1 \setminus B(x_1, 3r_1)$.

Note: $r(x_2) \leq M < \frac{8}{7} r(x_1) < \frac{8^2}{7^2} r_1 < \frac{4}{3} r_1$.

and $x_2 \notin B(x_1, 3r_1)$.

let $r_2 > \frac{7}{8} r(x_2)$ be such that $B(x_2, r_2) \in \mathcal{B}$.

Then $B(x_1, r_1) \cap B(x_2, r_2) = \emptyset$.

Continue inductively to find a disjoint collection

$\{ B(x_i, r_i) \}_{i=1}^{N_1}$ such that

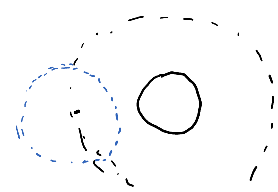
$$A_1 \subset \bigcup_{i=1}^{N_1} B(x_i, 3r_i)$$

(Important: This collection must be finite since there is a lower bound on r_i and $\mathbb{R}^d$ is finite dimensional)

Observe that for $x \in A_1$, $x \in B(x_i, 3r_i)$ for some i

and, as before, $r(x) < \frac{4}{3} r_i$, so

$$B(x, r(x)) \subset B(x_i, 5r_i)$$



(It is important that $\frac{4}{3} < 2$).

Step 2: Consider $A_2 = \{ x \in A \mid (\frac{7}{8})^2 \cdot M < r(x) \leq \frac{7}{8} \cdot M \}$.

Define $A_2' := \{ x \in A_2 \mid B(x, r(x)) \cap B(x_i, r_i) = \emptyset$ for all $i = 1, \dots, N_1 \}$.

Note: $$\bigcup_{x \in A_2 \setminus A_2'} B(x, r(x)) \subset \bigcup_{i=1}^{N_1} B(x_i, 5r_i)$$

For A_2' repeat the process for A_1 to find a

disjoint collection $_{N_2}$, $\{ B(x_i, r_i) \}_{i=N_1+1}^{N_2}$, s.t.

$i=N_1+1$,

$$A_2' \subset \bigcup_{i=N_1+1}^{N_2} B(x_i, 3r_i) \quad \text{and}$$

and $$\bigcup_{x\in A_2'} B(x, r(x)) \subset \bigcup_{i=N_1+1}^{N_2} B(x_i, 5r_i).$$

Furthermore, by the definition of A_2',

$$\{B(x_i, r_i)\}_{i=1}^{N_1} \cup \{B(x_i, r_i)\}_{i=N_1+1}^{N_2}$$

is a disjoint collection satisfying

$$\bigcup_{x\in A_1\cup A_2} B(x, r(x)) \subset \bigcup_{i=1}^{N_2} B(x_i, 5r_i).$$

With induction on A_k, we are done $\square$.

Q: Does A need to be bounded?

A: No, for A unbounded, let $A = \bigcup_{j=1}^{\infty} A_j$, A_j bounded

Perform $5r$ construction for A_1, Define $A_2' \subset A_2$ where maximal balls centered in A_2 do not intersect those constructed for A_1.
