Radon-Nikodym Differentiation

Lemma: Let μ and ν be finite positive measures.
Either $\nu \perp \mu$ or there exists $\varepsilon > 0$ and $E \in \mathcal{M}$ such that $\mu(E) > 0$ and $\nu \geq \varepsilon\mu$ on E.
(i.e. E is a positive set for $\nu - \varepsilon\mu$).

Pf:
Let $X = P_m \cup N_m$ be a Hahn decomposition for $\nu - \frac{1}{m}\mu$, $P_m \subset P_{m+1}$, $N_m \supset N_{m+1}$

$$\Rightarrow \quad \nu(N_m) \leq \frac{\mu(N_m)}{m} \leq \frac{\mu(X)}{m} \xrightarrow{m\to\infty} 0$$

$$\Rightarrow \quad \nu(P_m) \xrightarrow{m\to\infty} \nu(X) \Rightarrow \nu\Big|_{\bigcup_{m=1}^{\infty} P_m} = \nu.$$

Assuming ν and μ are not mutually singular, Therefore

$$\mu\left(\bigcup_{m=1}^{\infty} P_m\right) > 0 \quad \text{and } \exists\, m_0 \text{ s.t.}$$

$$\mu(P_{m_0}) > 0$$

Let $E = P_{m_0}$ and $\varepsilon = \frac{1}{m_0}$ $\square$.

Thm: (Lebesgue-Radon-Nikodym thm)
Let ν be a σ-finite signed measure & μ a σ-finite positive measur on $(X, \mathcal{M})$. $\exists!$ σ-finite signed measures λ, ρ on $(X,\mathcal{M})$ such that

$$\lambda \perp \mu, \quad \rho \ll \mu$$

and $\nu = \lambda + \rho$.

Moreover, $\exists g : X \to \overline{\mathbb{R}}$ measurable s.t. $d\rho = g\,d\mu$
g is unique up to sets of measure zero

Pf: First, suppose that ν and μ are <u>finite</u> positive measures. Define

$$\mathcal{F} = \left\{ f : X \to [0,\infty] \;\middle|\; \int_E f\,d\mu \le \nu(E) \quad \forall E \in \mathcal{M} \right\}$$

<u>Note:</u> If $f, g \in \mathcal{F}$, then $\max(f,g) \in \mathcal{F}$.

Let $M = \sup \left\{ \int f\,d\mu \;\middle|\; f \in \mathcal{F} \right\}$ and let $\{f_n\}_{n=1}^{\infty} \subset \mathcal{F}$ be such that $\int f_n\,d\mu \to M$.

Define $$g_m := \sup(f_1, \dots, f_m).$$

Then $g_m \le g_{m+1}$, $g_m \in \mathcal{F}$ so $\int g_m\,d\mu \le M$.

and $$\lim_{m\to\infty} \int g_m\,d\mu = M.$$

Let $g = \lim_{m\to\infty} g_m$. MCT implies $\int g\,d\mu = M$.

Let $$\rho(E) := \int_E g\,d\mu \quad \text{for all } E \in \mathcal{M}$$

and $$\lambda := \nu - \rho$$

Then $\nu = \lambda + \rho$, λ is a positive measure, and $\rho \ll \mu$.

It remains to show that $\lambda \perp \mu$

Suppose λ and μ are not mutually singular. Then by our lemma, $\exists\, \varepsilon > 0$ and $\exists\, E$, s.t. $\mu(E) > 0$ s.t.

$$\lambda \ge \varepsilon\mu \text{ on } E. \Rightarrow \varepsilon\mu|_E \le \lambda$$

$$\Rightarrow \varepsilon\mu|_E \le \nu - g\,d\mu \Rightarrow \varepsilon\mu|_E + g\,d\mu \le \nu$$

$\Rightarrow \quad (\varepsilon \chi_E + g)\, d\mu \le \nu. \quad \Rightarrow \quad \varepsilon \chi_E + g \in \mathcal{F}$

but $\quad \int \varepsilon \chi_E + g\, d\mu = \varepsilon\mu(E) + M > M$

which contradicts the definition of M ↯.

The <u>σ-finite case</u>: Assume ν and μ are <u>positive</u> σ-finite measures. $\exists \{X_j\}_{j=1}^{\infty} \subset \mathcal{M}$ s.t. $\bigcup_{j=1}^{\infty} X_j = X$

$X_j \cap X_i = \emptyset$ for $i \ne j$, and $\nu(X_j) < \infty$, $\mu(X_j) < \infty$ for all j

Then $\quad \nu = \sum \lambda_j + \sum \rho_j$, $\quad \sum \lambda_j \perp \mu$ and $\sum \rho_j \ll \mu$.

where λ_j and ρ_j are given by the finite case.

<u>The signed case</u>: Assume ν is a signed measure and let ν^- and ν^+ be the Hahn decomp of ν.

Then $\quad \nu = (\lambda^+ - \lambda^-) + (\rho^+ - \rho^-)$ $\quad \square$.

Some <u>Terminology</u>

For a signed measure, ν, the decomposition

$$\nu = \lambda + \rho$$

where $\quad \lambda \perp \mu$ and $\rho \ll \mu$

is called the <u>Lebesgue decomposition</u> of ν with respect to μ.

If $\nu \ll \mu$ then

$$\nu(E) = \int_E f\, d\mu \quad \text{for some } f \in L^1(\mu).$$

f is called the <u>Radon-Nikodym derivative</u> of ν with respect to μ. We denote

$$\frac{d\nu}{} = f$$

$$\frac{d\nu}{d\mu} = f.$$

and $$d\nu = \frac{d\nu}{d\mu}\, d\mu.$$

<u>Corollary</u>: Suppose that ν is a σ-finite signed measure and μ, λ are σ-finite measures on $(X, \mathcal{M})$ such that $\nu \ll \mu$ and $\mu \ll \lambda$

a.) If $g \in L^1(\nu)$, then $g \cdot \frac{d\nu}{d\mu} \in L^1(\mu)$ and

$$\int g\, d\nu = \int g \frac{d\nu}{d\mu}\, d\mu$$

b) $\nu \ll \lambda$ and $\frac{d\nu}{d\lambda} = \frac{d\nu}{d\mu}\frac{d\mu}{d\lambda}$ λ-a.e.

<u>Q</u>: In what sense is the Radon-Nikodym derivative a "derivative" in classical terms?

<u>Recall</u>: <u>Fundamental theorem of calculus</u>

If $F(x) = \int_0^x f(t)\, dt$ $F'(x) = f(x)$.

So if $\nu \ll \mu$, $\nu(E) = \int_E \frac{d\nu}{d\mu}\, d\mu$

does the following hold:

$$\frac{\nu(B(x,r))}{\mu(B(x,r))} \xrightarrow{r \to 0} \frac{d\nu}{d\mu}(x) \text{ ?}$$