Mutual Singularity

Def: Let μ and ν be signed measures. We say that μ and ν are mutually singular, denoted $\mu \perp \nu$, if $\exists E, F \subset X$ s.t. $E \cap F = \phi$, $E \cup F = X$,

$$\mu|_E = \mu \quad \text{and} \quad \nu|_F = \nu.$$

Examples:

- Let μ be any signed measure. Let $E \cap F = \phi$. Then $\mu|_E$ and $\mu|_F$ are mutually singular
- Consider $\delta_{\{x\}}$ and m then $E = \{x\}$, $F = \{x\}^c$ then $\delta_{\{x\}}|_E = \delta_{\{x\}}$ and $m|_F = m$.

Corollary (Jordan Decomposition)

If ν is a signed measure, there exist unique positive measures ν^+ and ν^- s.t.

$$\nu = \nu^+ - \nu^-$$

Moreover, ν^+ and ν^- are mutually singular.

Pf: Consider the Hahn decomposition of ν: P, N take $\nu^+ = \nu|_P$ $\nu^- = -\nu|_N$. $\square$

Def: (Total Variation)

If ν is a signed measure, define the total variation of ν by

$$|\nu| = \nu^+ + \nu^-$$

Absolute Continuity

Def: (Absolute Continuity)

Let ν be a signed measure and μ a positive measure on $(X, \mathcal{M})$. We say that ν is absolutely continuous w.r.t. μ, denoted $\nu \ll \mu$, if $\nu(E)=0$ whenever $\mu(E)=0$.

Examples:

- Let $E \in \mathcal{M}$ and μ be a measure. Then $\mu|_E \ll \mu$
- Let $f \in L^+$ be integrable and define $\nu(E) := \int_E f \, d\mu$ then $\nu \ll \mu$.