Q: Consider $A \times B \subset \mathbb{R}^2$ be Borel. Let $s, t \in [0,1]$

Suppose $\dim_{\mathcal{H}}(A) \leq s$ and $\dim_{\mathcal{H}}(B) \leq t$

$$\mathcal{H}^{s+t}(A \times B) = \mathcal{H}^s(A) \cdot \mathcal{H}^t(B) = (\mathcal{H}^s \times \mathcal{H}^t)(A \times B) \quad ?$$

A: No.

Let $\{n_k\}_{k=1}^{\infty} \subset \mathbb{N}$

and $\{m_k\}_{k=1}^{\infty} \subset \mathbb{N}$ s.t. $\{n_k\}_{k=1}^{\infty} \cap \{m_k\}_{k=1}^{\infty} = \phi$.

$\{n_k\}_{k=1}^{\infty} \cup \{m_k\}_{k=1}^{\infty} = \mathbb{N}$

Let $A = \{ x \in [0,1] \mid$ the n_k^{th} place of the decimal expansion of x is zero for all $k \}$.

$B = \{ x \in [0,1] \mid$ the m_k^{th} place of the decimal expansion of x is zero for all $k \}$.

Claim: For all $x \in [0,1]$, $\exists\, a \in A$, $b \in B$ s.t.

$x = a + b$.

Consider the function $F : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$F(x,y) = x+y$$

Then F is a linear function and

$[0,1] \subset F(A \times B)$. $\Rightarrow$ $\mathcal{H}^1(F(A \times B)) \geq 1$.

Linear functions do not increase Hausdorff dimension (HW)

and $\mathcal{H}^1(A \times B) \geq c\, \mathcal{H}^1(F(A \times B))$ for some $c > 0$.

However, one can choose $\{n_k\}$ and $\{m_k\}$ so that $\dim_{\mathcal{H}}(A)$ and $\dim_{\mathcal{H}}(B)$ are arbitrarily small (Exercise).

Differentiation of Measures

Signed Measures

Def: Let $(X, \mathcal{M})$ be a measurable space. A <u>signed measure</u> on $(X, \mathcal{M})$ is a function

$$\nu : \mathcal{M} \to [-\infty, \infty] \quad \text{s.t.}$$

- $\nu(\phi) = 0$
- ν assumes at most one of $-\infty$ or $+\infty$
- If $\{E_j\}_{j=1}^{\infty} \subset \mathcal{M}$ then

$$\nu\left(\bigcup_{j=1}^{\infty} E_j\right) = \sum_{j=1}^{\infty} \nu(E_j) \quad \text{where the}$$

convergence is absolute.

Notes: (1) Let $\{E_j\} \subset \mathcal{M}$. If $E_j \subset E_{j+1}$, then

$$\nu\left(\bigcup E_j\right) = \lim_{j \to \infty} \nu(E_j)$$

If $E_j \supset E_{j+1}$, then

$$\nu\left(\bigcap E_j\right) = \lim_{j \to \infty} \nu(E_j).$$

Def: (Null, Negative and Positive Sets)

a set $E \in \mathcal{M}$ is <u>positive</u> if for all $F \in \mathcal{M}$, $F \subset E$,

$$\nu(F) \geq 0$$

E is <u>negative</u> if $\forall F \in \mathcal{M}$, $F \subset E$, $\nu(F) \leq 0$

E is <u>null</u> if $\forall F \in \mathcal{M}$, $F \subset E$, $\nu(F) = 0$.

Hahn Decomposition Thm

Thm: If ν is a signed measure on $(X, \mathcal{M})$, there exist a positive set P and a negative set N for ν such that $P \cup N = X$, $P \cap N = \phi$. Moreover, P and N are unique up to null sets.

Pf: Suppose $\nu(E) < \infty$ for all $E \in \mathcal{M}$

Let $m = \sup\{\nu(F) \mid F \text{ positive}\}$.

$\Rightarrow \exists \{F_k\}_{k=1}^{\infty}$ F_k positive such that

$\lim_{k \to \infty} \nu(F_k) = m$

Let $P = \bigcup_{k=1}^{\infty} F_k$.

Then P is positive. Let $N = X \setminus P$

Claim: If $\infty > \nu(E) > 0$, then $\exists A \subset E$ s.t. $\nu(A) > 0$ and A is positive.

proof of claim: If E is not positive, Let $\varepsilon_1 \in [0,1]$ be the largest number s.t. $\exists F \subset E$ s.t. $\nu(F) < -\varepsilon_1$.

Let E_1 be such that $\nu(E_1) < -\varepsilon_1/2$

If $E \setminus E_1$ is not positive let $\varepsilon_2 \in [0,1]$ be s.t.

$\exists E_2 \subset E \setminus E_1$ s.t. $\nu(E_2) < -\varepsilon_2/2$

Inductively define ε_k and E_k and let $A = E \setminus \bigcup_{k=1}^{\infty} E_k$.

Then
$$\nu(E) = \nu(A) + \sum_{k=1}^{\infty} \nu(E_k) \Rightarrow \nu(A) > \nu(E) + \frac{1}{2}\sum_{k=1}^{\infty} \varepsilon_k$$

If $\sum_{k=1}^{\infty} \varepsilon_k = \infty$ then $\nu(A) = \infty$ and $\nu(\bigcup_k E_k) = -\infty$. which is a contradiction.

If $\sum_{k=1}^{\infty} \varepsilon_k < \infty$, then $\varepsilon_k \to 0$ which implies that there are no sets, $B \subset A \subset E$, such that $\nu(B) < 0$.

Otherwise we would contradict the maximality of ε_k.

The claim implies that N must be negative since P is

The claim implies that N must be negative since P is maximally positive $\square$.