Def: A metric space, (X, ρ), is separable if it contains a countable dense subset, D, i.e. $\overline{D} = X$.

Examples

1.) $(\mathbb{R}^d, \|\cdot\|)$ separable

2.) $(\mathbb{R}^d, \text{discrete})$ non-separable

3.) $\mathcal{C}(\mathbb{R})$ non-separable

4.) $\mathcal{C}([a,b])$ separable.

Proposition: A metric space (X, ρ) is separable iff $\exists$ a countable family of open sets $\{G_i\}_{i=1}^{\infty}$ such that any open set, $G \subset X$,

$$G = \bigcup_{G_i \in G} G_i$$

Lindelöf property.

$\boxed{\Rightarrow}$ Let $D = \{x_n\}_{n=1}^{\infty} \subset X$ be a set of dense points in X. Consider the collection of open balls

$$\left\{ B_{nm} := B(x_n, \tfrac{1}{m}) \right\}_{n,m=1}^{\infty}$$

Let $y \in G$, for m large enough

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$B(y, \frac{2}{m}) \subset G$ and since D is dense in X, $\exists\, x_n \in B(y, \frac{1}{2m})$

Then by Δ-ineq, $y \in B_{nm}$. Moreover,

$$B_{nm} = B(x_n, \tfrac{1}{m}) \subset B(y, \tfrac{2}{m}) \subset G$$

$$\Rightarrow \quad G \subset \bigcup_{B_{nm} \subset G} B_{nm} \subset G.$$

$\boxed{\Leftarrow}$ Let $\{G_i\}_{i=1}^{\infty}$ be the collection given.

For each G_i choose $x_i \in G_i$.

Let $D = \{x_i\}_{i=1}^{\infty}$. Now we want to show

$\overline{D} = X$.

Let F be a closed set containing D.

Then F^c is open, $F^c \not\supset G_i$ for all i and

$$F^c = \bigcup_{G_i \subset F^c} G_i$$

$$\Rightarrow F^c = \phi \Rightarrow F = X$$

$\Rightarrow \overline{D} = X$.

Complete Metric Space

Def: (Cauchy sequence)

Let $\{x_n\}_{n=1}^{\infty} \subset X$ be a sequence. We say that $\{x_n\}_{n=1}^{\infty}$ is Cauchy if for every $\varepsilon > 0$, $\exists N \in \mathbb{N}$

such that
$$n, m > N \implies \rho(x_n, x_m) < \varepsilon.$$

Def: A sequence $\{x_n\}_{n=1}^{\infty} \subset X$ converges to a point $x \in X$ if for all $\varepsilon > 0$, $\exists N \in \mathbb{N}$ such that
$$n > N \text{ implies } \rho(x_n, x) < \varepsilon$$

This is denoted by $\lim_{n \to \infty} x_n = x$.

Def: (Complete Metric Space).

A metric space (X, ρ) is complete if for every Cauchy sequence, $\{x_n\}_{n=1}^{\infty} \subset X$, there exists $x \in X$ s.t.
$$x_n \to x.$$

Ex: $(\mathbb{R}^d, \|\cdot - \cdot\|)$. is complete

Ex: $(\mathbb{Q}^d, \|\cdot - \cdot\|)$ is not complete.

Ex: (Polynomials on $[0,1]$, $\|\cdot - \cdot\|_\infty$) is not complete.

Ex: $((0,1), |\cdot - \cdot|)$.

Def: (Compact)

$E \subset X$ is a <u>compact</u> set if every open cover of E has a finite subcover; i.e. if $\{G_\alpha\}_{\alpha \in I}$ is a family of open sets satisfying $E \subset \bigcup_\alpha G_\alpha$ then $\exists \{G_i\}_{i=1}^n \subset \{G_\alpha\}_{\alpha \in I}$ such that

$$E \subset \bigcup_{i=1}^n G_i$$

<u>Examples:</u>

- Closed and bounded sets in Euclidean space are compact
- $\{ (a_n)_{n=1}^\infty \in \ell^2(\mathbb{N}) \mid (\sum_{n=1}^\infty |a_n|^2)^{1/2} \leq 1 \}$ is not compact so closed and bounded sets in $\ell^2(\mathbb{N})$ are not necessarily compact.