Lecture 19

Thm: Suppose $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ are σ-finite measure spaces. If $E \in \mathcal{M} \otimes \mathcal{N}$ then

(1) the functions defined by

$$x \mapsto \nu(E_x) \quad \text{and} \quad y \mapsto \mu(E^y) \quad \text{are measurable}$$

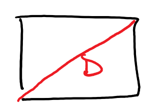
(2) $$\mu \times \nu(E) = \int \nu(E_x)\, d\mu(x) = \int \mu(E^y)\, d\nu(y)$$

(In particular, $\int \chi_E \, d(\mu\times\nu) = \iint \chi_E(x,y)\, d\mu(x)\, d\nu(y)$
$= \iint \chi_E(x,y)\, d\nu(y)\, d\mu(x)$)

Why do we need σ-finiteness?

Let $X = Y = [0,1]$, $\mathcal{M} = \mathcal{N} = \mathcal{B}_{[0,1]}$, $\mu = m$, $\nu =$ counting measure.

Let $D = \{(x,y) \in [0,1]\times[0,1] \mid x = y\}$.



Then $$\iint \chi_D(x,y)\, d\nu(y)\, d\mu(x) = \iint \chi_{\{x\}}(y)\, d\nu(y)\, d\mu(x) = \int 1\, d\mu(x) = \mu(X) = 1.$$

and

$$\int \chi_D(x,y)\, d(\mu\times\nu) = \mu\times\nu(D) = \inf\left\{\sum \pi(A_j \times B_j) \;\middle|\; D \subset \bigcup_j A_j\times B_j\right\}.$$
$$= \inf\left\{\sum \mu(A_j)\,\nu(B_j) \;\middle|\; D \subset \bigcup_j A_j \times B_j\right\}.$$

Claim: $\mu\times\nu(D) = \infty$.

and

$$\iint \chi_D(x,y)\, d\mu(x)\, d\nu(y) = \iint \chi_{\{y\}}(x)\, d\mu(x)\, d\nu(y) = \int 0 \, d\nu(y)$$
$$= 0.$$

Pf: The Finite case: Suppose $\mu(X) < \infty$ and $\nu(Y) < \infty$.

Consider the class of sets

Note: For $E = A \times B$, $A \in \mathcal{M}$, $B \in \mathcal{N}$

Then $$\nu(E_x) = \nu(B)\, \chi_A(x)$$

and $\quad \mu(E^y) = \mu(A)\chi_B(y)$ which are measurable.

Moreover, $\mu\times\nu(E) = \pi(A\times B) = \mu(A)\nu(B)$

and $\quad \int \nu(E_x)\,d\mu(x) = \int \nu(B)\chi_A(x)\,d\mu(x) = \nu(B)\mu(A)$

$\quad \int \mu(E^y)\,d\nu(y) = \int \mu(A)\chi_B(y)\,d\nu(y) = \mu(A)\nu(B)$.

$\Rightarrow$ (1) and (2) hold for finite unions of measurable rectangles

MCT implies that (1) and (2) hold for countable unions of measurable rectangles.

Another note Suppose (1) and (2) hold for a set $F \in \mathcal{M}\otimes\mathcal{N}$. Then F^c satisfies

$$\nu((F^c)_x) = \nu(Y) - \nu(F_x) \quad\text{and}\quad \mu((F^c)^y) = \mu(X) - \mu(F^y)$$

Since $\nu(Y)<\infty$ and $\mu(X)<\infty$

F^c satisfies (1) and (2).

Let $E\in\mathcal{M}\otimes\mathcal{N}$. By definition of $\mu\times\nu$, for all n

$\exists\, G_n = \bigcup_{j=1}^{\infty} A_j^n\times B_j^n$ s.t. $E\subset G_n$ and $\mu\times\nu(G_n\setminus E) < 1/2n$.

and $\exists\, H_n = \bigcup_{j=1}^{\infty} C_j^n\times D_j^n$ s.t. $E^c\subset H_n$ and $\mu\times\nu(E\setminus H_n^c) < \frac{1}{2n}$

$\Rightarrow \quad \mu\times\nu(G_n\setminus H_n^c) < \frac{1}{2n}$ and G_n, H_n satisfy (1) and (2).

Let $g_n(x) = \nu((G_n)_x)$ and $h_n(x) = \nu((H_n^c)_x)$

Monotonicity implies $\quad h_n(x) \le \nu(E_x) \le g_n(x)$ for all $x\in X$.

(1) and (2) imply $\quad \int g_n(x) - h_n(x)\,d\mu(x) = \mu\times\nu(G_n\setminus H_n^c) < \frac{1}{n}$

Thus g_n and h_n are Cauchy in $L^1(\mu)$

$\exists f$ s.t. $g_n \xrightarrow{L^1(\mu)} f$, $h_n \xrightarrow{L^1(\mu)} f$, f is μ-measurable

Moreover $f(x) = \nu(E_x)$ for μ-a.e. x. $\Rightarrow$ E satisfies (1).

E satisfies (2) from convergence theorems

$\square$

The Fubini - Tonelli Theorem

Thm: Suppose that (X, M, μ) and (Y, N, ν) are σ-finite measure spaces

i.) [Tonelli] If $f \in L^+(X \times Y)$ then the functions

$$(*) \quad g(x) = \int f_x(y)\, d\nu(y) \quad \text{and} \quad h(y) = \int f^y(x)\, d\mu(x)$$

satisfy

$g \in L^+(X)$, $h \in L^+(Y)$. Moreover,

$$(**) \quad \int f \, d(\mu \times \nu) = \int\left(\int f(x,y)\, d\mu(x)\right) d\nu(y) = \int\left(\int f(x,y)\, d\nu(y)\right) d\mu(x)$$

ii.) [Fubini] If $f \in L^1(\mu \times \nu)$ then $f_x \in L^1(\nu)$ μ-a.e. $x \in X$ and $f^y \in L^1(\mu)$ ν-a.e. $y \in Y$.
The functions g and h in $(*)$ satisfy $g \in L^1(\mu)$ and $h \in L^1(\nu)$. Furthermore, $(**)$ holds

---

pf: Monotone Convergence Theorem.

Examples

Let $\mathcal{L}^2$ be the Lebesgue measure on $\mathbb{R}^2$ and $\mathcal{L}^1$ be the Lebesgue measure on $\mathbb{R}$.

If E is $\mathcal{L}^2$-measurable then

$$\mathcal{L}^2(E) = \int \mathcal{L}^1(E_x)\, d\mathcal{L}^1(x).$$ Moreover, $\mathcal{L}^1 \times \mathcal{L}^1 = \mathcal{L}^2$.

Hausdorff Measure

Let $d \in \mathbb{N}$, $s \in (0, d]$.

For $E \subset \mathbb{R}^d$, define

$$\mathcal{H}^s_\delta(E) := \inf \left\{ \sum \operatorname{diam}(B_k)^s \;\middle|\; \begin{array}{l} B_k \text{ are open} \\ E \subset \bigcup_{k=1}^{\infty} B_k, \ \operatorname{diam}(B_k) < \delta \end{array} \right\}.$$

Then

<u>Def:</u> (Hausdorff Measure)

Given $d \in \mathbb{N}$, $s \in (0, \infty)$, $E \subset \mathbb{R}^d$, we define

$$\mathcal{H}^s(E) := \sup_{\delta > 0} \mathcal{H}^s_\delta(E)$$
$$= \lim_{\delta \to 0} \mathcal{H}^s_\delta(E).$$

<u>Notes:</u>
- $\mathcal{H}^s$ is a metric outer measure. (and thus a Borel outer measure) HW.
- If $s > d$, $\mathcal{H}^s(\mathbb{R}^d) = 0$.

<u>Def:</u> (Hausdorff Dimension).

Let $E \subset \mathbb{R}^d$ be a Borel set. Define the <u>Hausdorff dimension</u> of E by

$$\dim_{\mathcal{H}}(E) := \inf \{ s \in (0, \infty) \mid \mathcal{H}^s(E) = 0 \}.$$

<u>Q:</u> Consider $A \times B \subset \mathbb{R}^2$ be Borel. Let $s, t \in [0, 1]$

Suppose $\dim_{\mathcal{H}}(A) \leq s$ and $\dim_{\mathcal{H}}(B) \leq t$

$$\mathcal{H}^{s+t}(A \times B) = \mathcal{H}^s(A) \cdot \mathcal{H}^t(B) = (\mathcal{H}^s \times \mathcal{H}^t)(A \times B) \ ?$$

$$\mathcal{H}^{s+t}(A \times B) = \mathcal{H}^s(A) \cdot \mathcal{H}^t(B) = (\mathcal{H}^s \times \mathcal{H}^t)(A \times B) \quad ?$$