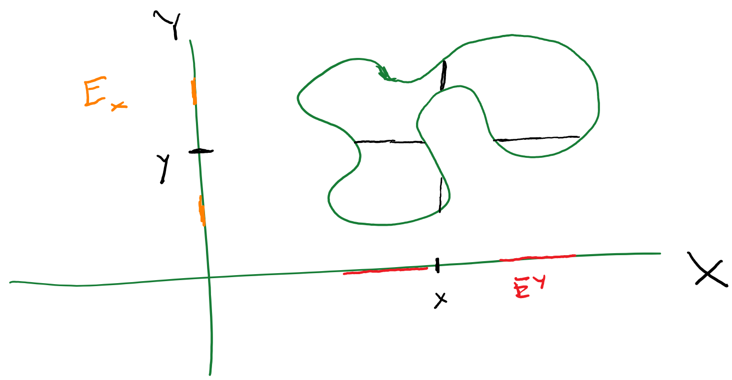
Measurable Functions

If $E \subset X \times Y$, for any $x \in X$, $y \in Y$ define

$$E_x = x \text{ section of } E := \{ y \in Y \mid (x,y) \in E \}$$

$$E^y = y \text{ section of } E := \{ x \in X \mid (x,y) \in E \}.$$

Picture



If $f : X \times Y \to \overline{\mathbb{R}}$ define

f_x = x-section of f : $f_x : Y \to \overline{\mathbb{R}}$, $f_x(y) = f(x,y)$.

f^y = y-section of f : $f^y : X \to \overline{\mathbb{R}}$, $f^y(x) = f(x,y)$

Proposition

i.) If $E \in \mathcal{M} \otimes \mathcal{N}$ then $E_x \in \mathcal{N}$ for all $x \in X$ and $E^y \in \mathcal{M}$ for all $y \in Y$.

ii.) If f is $\mathcal{M} \otimes \mathcal{N}$-measurable then f_x is $\mathcal{N}$-measurable for all $x \in X$ and f^y is $\mathcal{M}$-measurable for all $y \in Y$

Thm: Suppose (X, M, μ) and (Y, N, ν) are σ-finite measure spaces. If $E \in M \otimes N$ then

1. the functions defined by
$$x \mapsto \nu(E_x) \quad \text{and} \quad y \mapsto \mu(E^y) \quad \text{are measurable}$$

2. $$\mu \times \nu(E) = \int \nu(E_x)\, d\mu(x) = \int \mu(E^y)\, d\nu(y)$$

(In particular, $\int \chi_E \, d(\mu \times \nu) = \iint \chi_E(x,y)\, d\mu(x)\, d\nu(y)$
$= \iint \chi_E(x,y)\, d\nu(y)\, d\mu(x)$)

Pf: First, reduce to the case that (X, M, μ) and (Y, N, ν) are finite. Suppose the conclusions hold for finite measure spaces. Also suppose that (X, M, μ) and (Y, N, ν) are σ-finite.

(Y, N, ν) and (X, M, μ) are σ-finite implies there exist $\{Y_j\}_{j=1}^{\infty}$, $\{X_j\}_{j=1}^{\infty}$ such that $\mu(X_j) < \infty$, $\nu(Y_j)$, $Y_j \subset Y_{j+1}$, $X_j \subset X_{j+1}$ and $\bigcup_{j=1}^{\infty} X_j = X$, $\bigcup_{j=1}^{\infty} Y_j = Y$.

1. Let $E \in M \otimes N$.
$$\Rightarrow \nu(E_x) = \nu\left(\bigcup_{j=1}^{\infty} [E \cap (X_j \times Y_j)]_x\right)$$
$$= \nu\left(\bigcup_{j=1}^{\infty} E_x \cap Y_j\right) = \underbrace{\lim_{j \to \infty} \nu(E_x \cap Y_j)}_{\text{limit of measurable functions.}}.$$

2. Let $E \in M \otimes N$
$$\mu \times \nu(E) = \mu \times \nu\left(\bigcup_{j=1}^{\infty} E \cap (X_j \times Y_j)\right)$$
$$= \lim_{j \to \infty} \mu \times \nu\left(E \cap (X_j \times Y_j)\right) = \lim_{j \to \infty} \int \nu\left([E \cap X_j \times Y_j]_x\right) d\mu(x)$$
$$= \lim_{j \to \infty} \int \nu(E_x \cap Y_j)\, d\mu \overset{MCT}{=} \int \nu(E_x)\, d\mu(x).$$

The Finite case: Suppose $\mu(X) < \infty$ and $\nu(Y) < \infty$.

Consider the class of sets

$$\mathcal{C} := \{ E \in \mathcal{M} \otimes \mathcal{N} \mid E \text{ satisfies } ① \text{ and } ② \}.$$

<u>Goal:</u> Show that $\mathcal{R} \subset \mathcal{C}$ and $\mathcal{C}$ is a σ-algebra.

<u>Note:</u> For $E = A \times B$, $A \in \mathcal{M}$, $B \in \mathcal{N}$

Then $\nu(E_x) = \nu(B)\, \chi_A(x)$

and $\mu(E^y) = \mu(A)\, \chi_B(y)$ which are measurable.

Moreover, $\mu \times \nu(E) = \pi(A \times B) = \mu(A)\, \nu(B)$

and $\int \nu(E_x)\, d\mu(x) = \int \nu(B)\, \chi_A(x)\, d\mu(x) = \nu(B)\, \mu(A)$

$\int \mu(E^y)\, d\nu(y) = \int \mu(A)\, \chi_B(y)\, d\nu(y) = \mu(A)\, \nu(B)$.