Q: μ-a.e. convergence $\Rightarrow$ convergence in measure?

A: (1) with the assumptions of DCT, yes

(2) when $\mu(X) < \infty$.

Thm: (Egoroff's thm)

Suppose that $\mu(X) < \infty$, $\{f_n\}_{n=1}^{\infty}$ and f are measurable and $f_n \to f$ μ-a.e. Then for every $\varepsilon > 0$ $\exists E \subset X$ such that $\mu(E) < \varepsilon$ and $f_n \to f$ uniformly on E^c.

Pf: Let $F = \{x \in X \mid f_n(x) \not\to f(x)\}$, then by assumption $\mu(F) = 0$. Let $\varepsilon > 0$. For each $k, n \in \mathbb{N}$ define

$$E_{n,k} = \bigcup_{m=n}^{\infty} \{ |f_m - f| \geq \tfrac{1}{k} \}, \quad E_{n+1,k} \subset E_{n,k}$$

Claim: For each fixed k,

$$\bigcap_{n=1}^{\infty} E_{n,k} \subset F$$

The claim implies that $\lim_{n\to\infty} \mu(E_{n,k}) = 0$

Therefore, for each $k \in \mathbb{N}$, let n_k be large enough so that $\mu(E_{n_k,k}) < \varepsilon 2^{-k}$.

Define $E := \bigcup_{k=1}^{\infty} E_{n_k,k}$. Then $\mu(E) < \varepsilon$.

Claim: $f_n \to f$ uniformly on E^c

$\square$

Q: Does the type of convergence given by Egoroff's thm imply μ-a.e. convergence and convergence in measure?

Q: Can we replace the "$\mu(X) < \infty$" condition with "$|f_n| \le g$" for $g \in L^1(\mu)$?

Product Measures

In order to construct a coherent theory concerning the interchanging of a double integral, we need to start with product measures.

Let (X, M, μ) and (Y, N, ν) be complete measure spaces.

Let $\pi_X : X \times Y \to X$, $(x,y) \mapsto x$ and $\pi_Y : X \times Y \to Y$, $(x,y) \mapsto y$

Goal: construct a measure $\mu \times \nu$ on the σ-algebra $M \otimes N$

($M \otimes N$ = σ-algebra generated by

$$\mathcal{E} := \{ \pi_X^{-1}(E) \mid E \in M \} \cup \{ \pi_Y^{-1}(F) \mid F \in N \}.$$
)

Ex: $\mathbb{R} \times \mathbb{R}$

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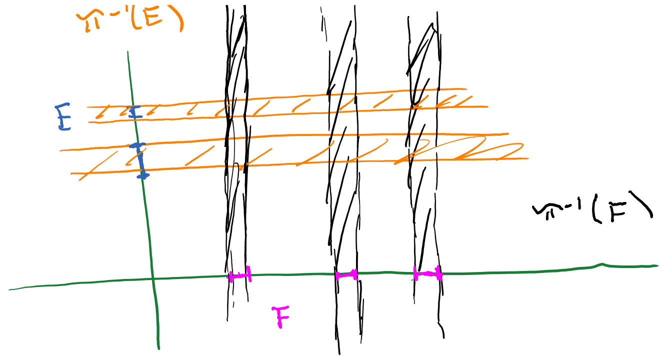

Let $\mathcal{R}$ be the collection of measurable rectangles

$$\mathcal{R} := \{ A\times B \mid A\in\mathcal{M},\ B\in\mathcal{N} \}$$

Some notes

- $\phi \in \mathcal{R}$
- $(A\times B)\cap(E\times F) = (A\cap E)\times(B\cap F)$
- $(A\times B)^c = A^c\times B \cup A\times B^c \cup A^c\times B^c$

If $R_+ := \{$ finite unions of disjoint rectangles $\}$ then R_+ is closed under finite unions and under complements.

Moreover $\mathcal{M}\otimes\mathcal{N} = \sigma(R_+)$.

Measuring unions of disjoint rectangles.

Let $E = \bigcup_{j=1}^{n} A_j\times B_j$ where $\{A_j\times B_j\}_{j=1}^{n}$ are disjoint.

Then for each j,

$$\chi_{A_j\times B_j}(x,y) = \chi_{A_j}(x)\chi_{B_j}(y)$$

Fix y: $\int \chi_{A_j\times B_j}(x,y)\,d\mu(x) = \int \chi_{A_j}(x)\chi_{B_j}(y)\,d\mu(x)$

$$= \begin{cases} 0 \cdot \int \chi_{A_j}(x)\, d\mu(x) & y \notin B_j \\ 1 \cdot \int \chi_{A_j}(x)\, d\mu(x) & y \in B_j. \end{cases}$$

$$= \begin{cases} 0 & y \notin B_j \\ \mu(A_j) & y \in B_j \end{cases}$$

$$\Rightarrow \quad \int \chi_{A_j \times B_j}(x,y)\, d\mu(x) = \mu(A_j)\, \chi_{B_j}(y)$$

<u>Integrate in</u> y.

$$\int \mu(A_j)\, \chi_{B_j}(y)\, d\nu(y) = \mu(A_j)\, \nu(B_j)$$

For E,

$$\int \chi_E(x,y)\, d\mu(x) = \int \sum_{j=1}^{n} \chi_{A_j}(x)\, \chi_{B_j}(y)\, d\mu(x)$$

$$\overset{MCT}{=} \sum_{j=1}^{n} \chi_{B_j}(y) \int \chi_{A_j}(x)\, d\mu(x)$$

$$= \sum_{j=1}^{n} \mu(A_j)\, \chi_{B_j}(y).$$

$$\Rightarrow \quad \iint \chi_E(x,y)\, d\mu(x)\, d\nu(y) = \sum_{j=1}^{n} \mu(A_j)\, \nu(B_j) \quad \text{by MCT}$$

<u>Define</u>: $\pi(A_j \times B_j) = \mu(A_j)\, \nu(B_j)$.

and for every $E \in R_+$, let $E = \bigcup_{j=1}^{n} A_j \times B_j$

and $\pi(E) := \sum_{j=1}^{n} \pi(A_j \times B_j)$.

<u>Define</u> For every set $E \subset X \times Y$

$$\pi_*(E) = \inf \left\{ \sum_{j=1}^{\infty} \pi(E_j) \;\middle|\; E \subset \bigcup E_j \quad E_j \in R_+ \right\}$$

Notes: (1) π_* is an outer measure

(2) $\pi_*\big|_{R_+} = \pi$

(3) If $E \in R_+$ then E is π_* measurable

i.e. $\pi_*(B) = \pi_*(E \cap B) + \pi_*(E^c \cap B)$ (Exercise)

(3) implies that $\sigma(R_+) \subset \{\pi_*$ measurable sets$\}$.

$\Rightarrow$ $M \otimes N \subset \{\pi_*$ measurable sets$\}$.

Define: $\mu \times \nu := \pi_*\big|_{M \otimes N}$.