Recall:

unif convergence $\Rightarrow$ pointwise convergence $\Rightarrow$ μ-a.e. convergence $\xRightarrow{\text{sometimes}}$ $L^1(\mu)$ convergence

(Convergence in measure)??

Lemma: $f_n \to f$ in $L^1(\mu)$ $\Rightarrow$ $f_n \to f$ in measure

pf: Let $\varepsilon > 0$

$$\mu(\{|f_n - f| \geq \varepsilon\}) \leq \varepsilon^{-1} \int |f_n - f| \, d\mu \to 0. \quad \square$$

Q: Convergence in measure $\Rightarrow$ convergence in $L^1(\mu)$?

Thm: Suppose that $\{f_n\}_{n=1}^\infty$ is Cauchy in measure. Then $\exists f$ s.t. $f_n \to f$ in measure and

$\exists \{f_{n_k}\}_{k=1}^\infty \subset \{f_n\}_{n=1}^\infty$ s.t. $f_{n_j} \to f$ μ-a.e.

Moreover, if $f_n \to g$ in measure then $f = g$ μ-a.e.

pf: $\{f_n\}_{n=1}^\infty$ Cauchy in measure implies that for all $\varepsilon > 0$

$$\lim_{n,m\to\infty} \mu(\{|f_n - f_m| \geq \varepsilon\}) \to 0.$$

Choose a subsequence $g_k = f_{n_k}$ s.t.

$$\mu(\{|g_k - g_{k+1}| \geq 2^{-j}\}) < 2^{-j}$$

Let $E_j := \{ |g_k - g_{k+1}| \geq 2^{-j} \}$

and $E := \{ x \in X \mid x \in E_j$ for infinitely many $j \}$.

Claim: $x \notin E \Rightarrow$ the sequence $\{g_k(x)\}_{k=1}^{\infty} \subset \mathbb{R}$ is Cauchy

The claim implies that for all $x \notin E$, $g_k(x) \to f(x)$, for some measurable f

Claim: $\mu(E) = 0$ (Borel - Cantelli)

$f_n \to f$ in measure since f_n is Cauchy

$$\mu(\{|f_n - f| \geq \varepsilon\}) \leq \underbrace{\mu(\{|f_n - g_j| \geq \varepsilon\})}_{\to 0} + \underbrace{\mu(\{|f - g_j| \geq \varepsilon\})}_{\to 0}$$

Uniqueness of convergence in measure

Similar argument

$$\mu(\{|f-g| \geq \varepsilon\}) \leq \underbrace{\mu(\{|f_n - g| \geq \varepsilon\})}_{\to 0} + \underbrace{\mu(\{|f_n - f| \geq \varepsilon\})}_{\to 0}$$

$\square$

Corollary:

Let $\{f_n\} \subset L^1(\mu)$ and assume that $f_n \to f$ in $L^1(\mu)$ then there exists a subsequence $\{f_{n_k}\}$ s.t. $f_{n_k} \to f$ μ-a.e.