$L^1(\mu)$ as a complete metric Space

We first introduce a normed vector space.

Def: (Norm)
Given a vector space, X, a norm on X is a function

$$\|\cdot\| : X \to [0,\infty) \quad \text{satisfying}$$

i.) $\|x\|=0$ iff $x=0$
ii.) $\|\lambda x\| = |\lambda|\,\|x\| \quad \forall x\in X, \ \lambda\in\mathbb{C}$
iii.) $\|x+y\| \le \|x\|+\|y\|$ (Triangle inequality)

Note: A norm produces a metric on X defined by
$$\rho(x,y) = \|x-y\|.$$

Def: (Convergence and Absolute Convergence of Series)
Let $\{x_n\}_{n=1}^{\infty} \subset X$
① We say that $\sum_{n=1}^{\infty} x_n$ converges to x if $\lim_{N\to\infty} \sum_{n=1}^{N} x_n = x$.
② We say that $\sum_{n=1}^{\infty} x_n$ absolutely converges if $\sum_{n=1}^{\infty} \|x_n\| < \infty$

Thm: A normed space, $(X, \|\cdot\|)$, is complete iff every absolutely convergent series converges in X.

Motivation? Think of a normed space where every absolutely convergent series doesn't necessarily converge.

Pf: Δ-ineq.

Thm (Riesz-Fisher)

$L^1(\mu)$ is a complete metric space

pf: Follows directly from the previous two theorems.

Modes of Convergence of integrable functions

Recall: (1) Uniform convergence

(2) Pointwise Convergence

(3) Pointwise μ-a.e. convergence (μ-a.e. convergence)

(4) Convergence in $L^1(\mu)$ (as a metric space).

$$1 \underset{\text{easy}}{\overset{\not\Leftarrow}{\Rightarrow}} 2 \underset{\text{easy}}{\overset{\not\Leftarrow}{\Rightarrow}} 3 \underset{\text{In some cases using MCT and DCT}}{\overset{?\Leftarrow}{\Longrightarrow}} 4$$

Consider another form of convergence

Def: (Convergence in Measure)

Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of measurable functions.

We say that f_n converges in measure to a measurable function f if

for all $\varepsilon > 0$

$$\mu(\{x \in X \mid |f_n(x) - f(x)| \geq \varepsilon\}) \xrightarrow{n\to\infty} 0.$$

Ex: Let $\{q_n\} \subset [0,1]$ be an enumeration of the rationals.

Define a sequence of sets

$$E_n = B(q_n, \tfrac{1}{n}).$$

and define $f_n(x) = a_n \chi_{E_n}$ for some sequence $\{a_n\}_{n=1}^{\infty} \subset \mathbb{R}$.

Then $f \xrightarrow{m} f = 0$

Then $f_n \xrightarrow{m} f \equiv 0$