Lebesgue Dominated Convergence Theorem

Thm: Let (X, M, μ) be a complete measure space.

Let $\{f_n\}_{n=1}^{\infty} \subset L^1(\mu)$ be such that

i.) $f_n \to f$ μ-a.e.

ii.) $\exists\, g : X \to [0,\infty]$, $g \in L^1(\mu)$ s.t. $|f_n| \le g$ $\forall n$ μ-a.e.

Then $f \in L^1(\mu)$ and $\int f\, d\mu = \lim_{n\to\infty} \int f_n\, d\mu$.

Moreover, $\int |f_n - f|\, d\mu \to 0$ as $n \to \infty$.

(I.e. $f_n \to f$ in $L^1(\mu)$).

pf: (Hard way (Easy way is in text)).

1st. Assume $\mu(X) = A < \infty$ and $g \le M$ for some $A, M > 0$.

We want to show that $\int |f_n - f|\, d\mu \to 0$.

Let $\varepsilon > 0$ and let $\delta = \frac{\varepsilon}{2A}$.

Define $E_N = \{x \in X \mid \sup_{n \ge N} |f_n(x) - f(x)| > \delta\}$.

Claim: $E_N \supset E_{N+1}$ and $\bigcap E_N \subset F$ where $\mu(F) = 0$

$\Rightarrow$ for all $n \ge N$, $\int |f_n - f(x)|\, d\mu$

$$= \int_{E_N} |f_n - f|\, d\mu + \int_{E_N^c} |f_n - f|\, d\mu$$

$$\le M\mu(E_N) + \delta\,\mu(E_n^c)$$

$$< M\mu(E_N) + \delta A$$

Not $\lim_{N\to\infty} \mu(E_N) = \mu(\bigcap E_N) = 0$.

$\Rightarrow \sup_{n\geq N} \int |f_n - f|\, d\mu < M\cdot\frac{\varepsilon}{2M} + \frac{\varepsilon}{2A}\cdot A$ for N large

$\underline{2^{nd}}$ Reduce to first case.

$\underline{\text{Note}}$: $|f_n - f| \leq 2g$

Let $\varepsilon > 0$.

By Chebyshev, $\mu(\{x\in X \mid 2g \geq K\}) \leq \frac{1}{K}\int 2g$

Also, since $2g \in L^1(\mu)$ there exists $\delta > 0$ s.t.

$\mu(E) < \delta$ implies $\int_E 2g\, d\mu < \varepsilon/3$.

$\Rightarrow$ For K large enough

$\int_{\{2g\geq K\}} 2g\, d\mu < \varepsilon/3 \Rightarrow \int_{\{2g\geq K\}} |f_n - f|\, d\mu < \varepsilon/3$.

On the other hand, the proposition implies that

There exists $B \subset X$ such that $\mu(B) < \infty$

and $\int_{B^c} 2g\, d\mu < \varepsilon/3 \Rightarrow \int_{B^c} |f_n - f|\, d\mu < \varepsilon/3$.

(The proposition needs μ to be σ-finite. Why is this ok?)

Now it suffices to show that for n large enough

$\int_{X - (B^c \cup \{2g\geq K\})} |f_n - f|\, d\mu < \varepsilon/3$. But this follows from the

first case. $\square$

Q: $f: \mathbb{R} \to [0,\infty)$, $f \in L^1(m) \Rightarrow \lim_{x\to\infty} f(x) = 0$.

a.) What if f is continuous?

b.) What if f is uniformly continuous?

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Thm: Suppose $\{f_j\}_{j=1}^{\infty} \subset L^1(\mu)$ and $\sum_{j=1}^{\infty} \int |f_j|\, d\mu < \infty$.

then $\sum_{j=1}^{\infty} f_j$ converges μ-a.e. and in $L^1(\mu)$

to a function, $f \in L^1(\mu)$.

Moreover,
$$\int \sum_{j=1}^{\infty} f_j \, d\mu = \sum_{j=1}^{\infty} \int f_j \, d\mu$$

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pf: $\underline{\mu\text{-a.e.}}$ convergence

MCT $\Rightarrow$ $\sum_{j\geq 1} \int |f_j| \, d\mu = \int \sum_{j=1}^{\infty} |f_j| \, d\mu < \infty$

$\xRightarrow{\text{(by HW2)}}$ $\mu\left(\left\{ \sum |f_j| = \infty \right\}\right) = 0.$

$\Rightarrow$ $\sum f_j(x)$ converges absolutely for μ-a.e. $x \in X$.

$\underline{L^1 \text{ convergence and } \int \sum_{j=1}^{\infty} f_j \, d\mu = \sum_{j=1}^{\infty} \int f_j \, d\mu}$

For all N, $\left|\sum_{j=1}^{N} f_j\right| \leq \sum_{j=1}^{\infty} |f_j| \in L^1(\mu)$

DCT $\Rightarrow$ L^1 convergence and $\int \sum f_j \, d\mu = \sum_{j=1} \int f_j \, d\mu$. $\square$

This allows for the begin to construct a suitable setting to study Fourier Series of integrable functions

Consider complex-valued functions $f : \mathbb{R}/\mathbb{Z} \to \mathbb{C}$,

Let $e(x) := e^{2\pi i x}$ and consider the family of series

$$A(\mathbb{T}) = \left\{ \sum a_n e(nx) \;\middle|\; \sum |a_n| < \infty \right\}.$$

$$\mathbb{A}(\mathbb{T}) = \left\{ \sum_{n \in \mathbb{Z}} a_n e(nx) \;\middle|\; \sum |a_n| < \infty \right\}.$$

If we define the Fourier transform,

$$\hat{f}(n) := \int_{\mathbb{T}} f(x)\, e(nx)\, dm(x) \quad \text{for all } n \in \mathbb{Z}.$$

By the previous thm, if $f(x) = \sum a_n e(nx) \in \mathbb{A}$

then $\hat{f}(n) = a_n$.