Def: (Integrable Function)

Let $f : X \to \overline{\mathbb{R}}$ be measurable, we say that integrable with respect to μ if

$$\int |f|\, d\mu = \int f^+ d\mu + \int f^- \, d\mu < \infty.$$

Prop: If f, g are measurable, $\int |f|\, d\mu < \infty$, $\int |g|\, d\mu < \infty$ then for $\alpha, \beta \in \mathbb{R}$

$$\int (\alpha f + \beta g)\, d\mu = \alpha \int f\, d\mu + \beta \int g\, d\mu$$

pf: Exercise (Note: We've already shown this for $\alpha, \beta \geq 0$, $f, g \geq 0$).

Prop: If f is measurable

$$\left| \int f\, d\mu \right| \leq \int |f|\, d\mu.$$

pf: Δ- inequality.

Prop: If f, g are measurable, $\int |f|\, d\mu < \infty$, $\int |g|\, d\mu < \infty$ then the following are equivalent.

i.) $\int_E f\, d\mu = \int_E g\, d\mu \quad \forall E \in \mathcal{M}$.

ii.) $\int |f-g|\, d\mu = 0$

iii.) $f = g \quad \mu$-a.e.

pf: Δ-inequality and previous prop gives (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii).

$L^1(\mu)$

An Equivalence Class.

If $f, g : X \to \overline{\mathbb{R}}$ are measurable, define

$f \sim g$ iff $f = g$ μ-a.e.

Exercise: Show that this is an equivalence class.

$[f] := \{ g : X \to \overline{\mathbb{R}} : f = g \ \mu\text{-a.e.} \}$

$$\boxed{L^1(\mu) := \{ [f] : \int |f| \, d\mu < \infty \}.}$$

if $g \in [f]$ $\quad f = g$ μ-a.e. $\Rightarrow \int |f| \, d\mu = \int |g| \, d\mu.$

On $L^1(\mu)$, we define the following metric

$\rho : L^1(\mu) \times L^1(\mu) \to [0, \infty)$

$\rho(f, g) := \int |f - g| \, d\mu =: \|f - g\|_{L^1(\mu)}$

Then $(L^1(\mu), \|\cdot - \cdot\|_{L^1(\mu)})$ is a metric space.

(Q: Is $(L^1(\mu), \|\cdot - \cdot\|_{L^1(\mu)})$ complete?).

Prop: Let $(X, \mathcal{M}, \mu)$ be a complete measure space.
If $f \in L^1(\mu)$ then

1. $\forall \varepsilon > 0 \ \exists \delta > 0$ s.t. $\int_E |f| < \varepsilon$ whenever $\mu(E) < \delta$

2. If μ is σ-finite, then $\forall \varepsilon > 0$, $\exists B \in \mathcal{M}$ s.t. $\mu(B) < \infty$ and $\int_{B^c} |f| \, d\mu < \varepsilon$.

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pf: 1. Let $\varepsilon > 0$, and $E_n := \{ x \in X \mid n \le |f| \le n+1 \}$.

If $f_n := f \cdot \chi_{E_n}$, then by MCT

$$\sum \int |f_n| \, d\mu = \int |f| \, d\mu < \infty.$$

$\Rightarrow \exists N$ s.t. $\sum_{n \geq N} \int |f_n| \, d\mu < \varepsilon/2$

and $\left(\bigcup_{n \geq N} E_n\right)^c = \{x \in X \mid |f| < N\}$.

Let $F := \left(\bigcup_{n \geq N} E_n\right)^c$, then

$$\int |f| \, d\mu = \int_F |f| \, d\mu + \int_{F^c} |f| \, d\mu < \int_F |f| \, d\mu + \varepsilon/2.$$

and for any $E \subset \mathcal{M}$,

$$\int_E |f| \, d\mu = \int_{E \cap F} |f| \, d\mu + \int_{E \cap F^c} |f| \, d\mu < N\mu(E \cap F) + \varepsilon/2.$$
$$\leq N\mu(E) + \varepsilon/2.$$

so let $\delta = \frac{\varepsilon}{2N}$.

(2) μ σ-finite implies that

$X = \bigcup_{j=1}^{\infty} X_j$ where $\mu(X_j) < \infty$ for all j.

Let $\varepsilon > 0$, and let $E_N := \bigcup_{j=1}^{N} X_j$ and

$$f_N := f \cdot \chi_{E_N}.$$

Then $|f_N| \leq |f|$ $\forall N$ and by MCT

$$\lim_{N \to \infty} \int |f_N| = \int |f| \, d\mu.$$

$\Rightarrow \exists M$ s.t. $N > M$ implies $\int |f_N| \, d\mu - \int |f| \, d\mu < \varepsilon$

$$\Rightarrow \int |f_M| - |f| \, d\mu < \varepsilon$$

$$\Rightarrow \int_{E_N^c} |f| \, d\mu < \varepsilon. \quad \square$$

Lebesgue Dominated Convergence Theorem.

Thm: Let $(X, \mathcal{M}, \mu)$ be a complete measure space.

Let $\{f_n\}_{n=1}^{\infty} \subset L^1(\mu)$ be such that

i.) $f_n \to f$ μ-a.e.

ii.) $\exists\, g : X \to [0, \infty]$, $g \in L^1(\mu)$ s.t. $|f_n| \le g$ $\forall n$ μ-a.e.

Then $f \in L^1(\mu)$ and $\int f \, d\mu = \lim_{n\to\infty} \int f_n \, d\mu$.

Moreover, $\int |f_n - f| \, d\mu \to 0$ as $n \to \infty$.

(I.e. $f_n \to f$ in $L^1(\mu)$).

Nonexamples

- $f_n(x) = n\, \chi_{(0,\frac{1}{n})}(x)$
- $f_n(x) = \chi_{(n, n+1)}(x)$