Thm (Monotone Convergence Theorem)

If $\{f_n\}_{n=1}^{\infty} \subset L^+$ is a sequence such that

$$0 \le f_n \le f_{n+1} \quad \text{and}$$

$$f := \lim_{n\to\infty} f_n$$

Then

$$\int f \, d\mu = \int \lim_{n\to\infty} f_n \, d\mu = \lim_{n\to\infty} \int f_n \, d\mu$$

Pf: Recall: $0 \le f_n \le f_{n+1} \le f$

$$\Rightarrow \int f_n \, d\mu \le \int f_{n+1} \, d\mu \le \int f \, d\mu.$$

$$\Rightarrow \lim_{n\to\infty} \int f_n \, d\mu \le \int f \, d\mu$$

So it suffices to show that $\lim_{n\to\infty} \int f_n \, d\mu \ge \int f \, d\mu$

Let $\phi : X \to [0,\infty)$ be a simple function.

satisfying $\phi \le f$.

Let $\alpha \in (0,1)$.

Note: For every $\alpha \in (0,1)$, $\nu : \mathcal{M} \to [0,\infty]$, defined by $\nu(E) := \int_E \alpha \phi \, d\mu$ is a measure.

Note: Since $\alpha < 1$, $\lim_{n\to\infty} f_n(x) > \alpha \phi(x)$ for all $x \in X$

$\Rightarrow \forall x \in X, \ \exists N_x$ s.t. $f_n(x) > \alpha \phi(x)$ for all $n \ge N$.

For each $n \in \mathbb{N}$, define

$$E_n := \{ x \in X \mid f_n(x) > \alpha \phi(x) \}.$$

Claim: $X = \bigcup_{n=1}^{\infty} E_n$ and $E_n \subset E_{n+1}$ (follows from the second note)

Now, for all n

$$\int_{E_n} \alpha \phi \, d\mu \le \int_{E_n} f_n d\mu \le \int_X f_n \, d\mu$$

$$\Rightarrow \quad \int_{E_n} \alpha \phi \, d\mu \le \lim_{n\to\infty} \int_X f_n \, d\mu.$$

If $\nu_\alpha(E) := \int_E \alpha \phi \, d\mu$, then $\nu_\alpha(E_n) \le \lim_{n\to\infty} \int_X f_n \, d\mu.$

which implies $\lim_{n\to\infty} \nu_\alpha(E_n) \le \lim_{n\to\infty} \int_X f_n \, d\mu.$

Since ν_α is a measure,

$$\lim_{n\to\infty} \nu_\alpha(E_n) = \nu_\alpha\left(\bigcup_{n=1}^{\infty} E_n\right) = \nu_\alpha(X)$$

$$\Rightarrow \quad \nu_\alpha(X) \le \lim_{n\to\infty} \int_X f_n d\mu \iff \int \alpha\phi \, d\mu \le \lim_{n\to\infty} \int f_n d\mu.$$

$$\Rightarrow \quad \int \phi \, d\mu \le \lim_{n\to\infty} \int f_n \, d\mu.$$

$$\Rightarrow \quad \int f \, d\mu = \sup_{\phi \le f} \int \phi \, d\mu \le \lim_{n\to\infty} \int f_n d\mu. \quad \square$$

Similar to Chebyshev's inequality argument

$$\mu(\{x\in X \mid f(x) > \alpha\}) = \int \chi_{\{f>\alpha\}} \, d\mu = \frac{1}{\alpha} \int \alpha \chi_{\{f>\alpha\}} \, d\mu \le \frac{1}{\alpha} \int f \, d\mu.$$

The following Lemma can now easily be proven using limits of monotonic sequences.

Lemma: Let $f, g \in L^+$, then

$$\int f + g \, d\mu = \int f \, d\mu + \int g \, d\mu$$

$\int f+g \, d\mu = \int f \, d\mu + \int g \, d\mu$

Thm: Let $\{f_n\}_{n=1}^{\infty} \subset L^+$, If $f = \sum_{n=1}^{\infty} f_n$ then

$$\int f \, d\mu = \sum_{n=1}^{\infty} \int f_n \, d\mu$$

Prop: Let $f \in L^+$.

$\int_X f \, d\mu = 0$ if and only if $f = 0$ μ-a.e.

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pf: $(\Rightarrow)$ Observe that
$$\{x \in X \mid f(x) \neq 0\} = \bigcup_{n \in \mathbb{N}} \{x \in X \mid f(x) > \tfrac{1}{n}\}.$$

Chebyshev $\Rightarrow$ $\mu(\{x \in X \mid f(x) > \tfrac{1}{n}\}) \le n \int_X f \, d\mu = 0.$

$$\Rightarrow \mu(\{x \in X \mid f(x) \neq 0\}) \le \sum \mu(\{x \in X \mid f(x) > \tfrac{1}{n}\}) = 0.$$

$(\Leftarrow)$ Let $0 \le \varphi \le f$ where φ is a simple function.

Let $E = \{x \in X \mid f(x) = 0\}$. Then $\mu(E^c) = 0$.

$\varphi = \sum \alpha_i \chi_{A_i}$. If $E \cap A_i \neq \emptyset$, then $\alpha_i = 0$.

$$\Rightarrow \int \varphi \, d\mu = \int_E \varphi \, d\mu + \int_{E^c} \varphi \, d\mu$$

$$= \sum \alpha_i \mu(E \cap A_i) + \sum \alpha_i \underbrace{\mu(E^c \cap A_i)}_{=0} = 0.$$

$= 0$ $\square$

<u>Corollary</u> Let $\{f_n\} \subset L^+$, $f \in L^+$ and

- $f_n \le f_{n+1}$ μ-a.e.
- $f(x) = \lim_{n\to\infty} f_n(x)$ μ-a.e.

Then $$\lim_{n\to\infty} \int f_n \, d\mu = \int f \, d\mu$$

Fatou's Lemma

<u>Lem:</u> If $\{f_n\}_{n=1}^{\infty} \subset L^+$, then

$$\int \left(\liminf_{n\to\infty} f_n\right) d\mu \le \liminf_{n\to\infty} \int f_n \, d\mu.$$

<u>pf:</u> Let $g_n := \inf_{k \ge n} f_k$, then $\liminf_{n\to\infty} f_n = \lim_{n\to\infty} g_n$

and $g_n \le g_{n+1}$

$\Rightarrow$ (MCT) $\int \liminf_{n\to\infty} f_n \, d\mu = \int \lim_{n\to\infty} g_n \, d\mu = \lim_{n\to\infty} \int g_n \, d\mu.$

Note: $g_n \le f_k$ for all $k \ge n$ so

$$\int g_n \, d\mu \le \int f_k \, d\mu$$

$\Rightarrow$ $\lim_{n\to\infty} \int g_n \, d\mu \le \liminf_{n\to\infty} \int f_n \, d\mu$ $\square$.

Integration & Real-Valued Functions

Integration & Real-Valued Functions

Let (X, M, μ) be a complete measure space. Let $f : X \to \overline{\mathbb{R}}$ be measurable.

Define
$$f^+ := \max\{0, f\}.$$
$$f^- := \max\{0, -f\}.$$

If $\int f^+ \, d\mu < \infty$ or $\int f^- \, d\mu < \infty$, define
$$\int f \, d\mu = \int f^+ d\mu - \int f^- d\mu.$$

Def: (Integrable Function)
Let $f : X \to \overline{\mathbb{R}}$ be measurable, we say that f is integrable with respect to μ if
$$\int |f| \, d\mu = \int f^+ d\mu + \int f^- d\mu < \infty.$$