Def: A simple function $f: X \to \mathbb{R}$ is a finite linear combination of characteristic functions of measurable sets with real coefficients

$$f(x) = \sum_{i=1}^{m} \alpha_i \chi_{A_i} \qquad A_i \in \mathcal{M}.$$

We say that f is in <u>standard representation</u> if

$$f = \sum_{i=1}^{n} \beta_i \chi_{E_i} \qquad \beta_i \neq \beta_j \quad , \quad E_i \cap E_j = \phi.$$

Thm: Let $(X, \mathcal{M})$ be a measurable space

If $f: X \to [0, \infty]$ is measurable, then there is a sequence $\{\phi_n\}_{n=1}^{\infty}$ of simple functions s.t.

$$0 \le \phi_1 \le \phi_2 \le \cdots \le \phi_n \le \cdots \le f.$$

with $\phi_n \to f$ pointwise.

Pf: Instead of approximating a function by decomposing its domain, decompose the co-domain.

Prop: Let $(X, \mathcal{M}, \mu)$ be a measure space. The following implications iff μ is complete.

1.) If $f: X \to \overline{\mathbb{R}}$ is measurable and $f = g$ μ-a.e. then g is μ-measurable.

1.) If $f: X \to \mathbb{R}$ is ... and $f = g$ μ-a.e.
then g is μ-measurable.

(μ-a.e. = μ-almost everywhere. Which means that $\{f \neq g\} \in M$ and $\mu(\{f \neq g\}) = 0$.)

2.) If $\{f_n\}$ is a sequence of measurable functions $f_n \to f$ μ-a.e. then f is measurable.

Integration

Let (X, M, μ) be a complete measure space.

Denote the class of measurable functions by

$$L^+ := \{ f : X \to [0, \infty) \text{ measurable} \}.$$

Def: If $\phi = \sum_{i=1}^n \alpha_i \chi_{E_i}$ is a simple function with standard representation we define the integral of ϕ w.r.t. μ as.

$$\int_X \phi \, d\mu = \int \phi \, d\mu = \sum_{i=1}^n \alpha_i \mu(E_i)$$

For $A \in M$,

$$\int_A \phi \, d\mu = \int_X \phi \cdot \chi_A \, d\mu = \int_X \sum_{i=1}^n \alpha_i \chi_{E_i} \cdot \chi_A \, d\mu$$

$$= \int_X \sum_{i=1}^n \alpha_i \chi_{E_i \cap A} \, d\mu$$

$$= \sum_{i=1}^n \ldots$$

$$= \sum_{i=1}^{n} \alpha_i \, \mu(E_i \cap A)$$

Prop: Let ϕ and ψ be simple functions in L^+

i.) If $c \geq 0$ $\int c\phi \, d\mu = c \int \phi \, d\mu$

ii.) $\int (\phi + \psi) \, d\mu = \int \phi \, d\mu + \int \psi \, d\mu$

iii.) If $0 \leq \phi \leq \psi$

$$\int \phi \, d\mu \leq \int \psi \, d\mu$$

iv.) If $\nu(A) = \int_A \phi \, d\mu$, for all $A \in \mathcal{M}$ then

ν is a measure on $\mathcal{M}$.

Def: If $f \in L^+$ define

$$\int f \, d\mu := \sup\left\{ \int \phi \, d\mu : 0 \leq \phi \leq f, \ \phi \text{ is a simple function} \right\}.$$

Notes: i.) if $f, g \in L^+$, $f \leq g$ then

$$\int f \, d\mu \leq \int g \, d\mu$$

ii.) If $c \geq 0$ $\int cf \, d\mu = c \int f \, d\mu$.

What about sequences & functions?

Very Important Theorem:

Thm (Monotone Convergence Theorem)

If $\{f_n\}_{n=1}^{\infty} \subset L^+$ is a sequence such that

$$0 \leq f_n \leq f_{n+1} \quad \text{and}$$

$$f := \lim_{n\to\infty} f_n$$

Then

$$\int f \, d\mu = \int \lim_{n\to\infty} f_n \, d\mu = \lim_{n\to\infty} \int f_n \, d\mu$$