Def: (Measurable Functions)

Let (M, X) and (N, Y) be measurable spaces

a function $f: X \to Y$ is (M, N) measurable

if $f^{-1}(E) \in M$ for all $E \in N$.

Ex: Let $f: (X, M) \to \mathbb{R}$, Let $E \in M$, then

χ_E is measurable

Proposition If N is a σ-algebra generated by a collection $\mathcal{E}$, then $f: X \to Y$ is (M, N)-measurable iff $f^{-1}(E) \in M$ for all $E \in \mathcal{E}$.

Corollary: If X and Y are metric spaces, every continuous function $f: X \to Y$ is $(\mathcal{B}_X, \mathcal{B}_Y)$-measurable

Real-valued functions

Common Terminology

1. Let $f: (X, M) \to \mathbb{R}$. f is referred to simply as measurable if f is $(M, \mathcal{B}_{\mathbb{R}})$-measurable

2. $f: \mathbb{R} \to \mathbb{R}$ is Lebesgue measurable if it is $(\mathcal{L}, \mathcal{B}_{\mathbb{R}})$ measurable and Borel measurable if it is $(\mathcal{B}_{\mathbb{R}}, \mathcal{B}_{\mathbb{R}})$ measurable

- If $f, g: \mathbb{R} \to \mathbb{R}$ are both Lebesgue measurable, then $f \circ g$ is not necessarily Lebesgue measurable.

<u>Proposition</u> Let $(X, \mathcal{M})$ be a measurable space and $f: X \to \mathbb{R}$. The following are equivalent

i.) f is measurable

ii.) $f^{-1}((a,\infty)) \in \mathcal{M} \quad \forall a \in \mathbb{R}$

iii.) $f^{-1}((-\infty,a)) \in \mathcal{M} \quad \forall a \in \mathbb{R}$

iv.) $f^{-1}((-\infty,a]) \in \mathcal{M} \quad \forall a \in \mathbb{R}$

v.) $f^{-1}([a,\infty)) \in \mathcal{M} \quad \forall a \in \mathbb{R}$

vi.) $f^{-1}((a,b)) \in \mathcal{M} \quad \forall a,b \in \mathbb{R}$.

<u>Proposition</u>: Let $(X, \mathcal{M})$ be a measurable space, $f, g: X \to \mathbb{R}$ measurable, $c \in \mathbb{R}$.

Then $c \cdot f$, $f+g$ and $f \cdot g$ are measurable

<u>pf</u>: Show that $h^{-1}((a,\infty)) \in \mathcal{M}$ for all $a \in \mathbb{R}$

① <u>$c \cdot f$</u>.

Note for $c > 0$

$$(c \cdot f)^{-1}((a,\infty)) = \{x \in X \mid c \cdot f(x) > a\}$$
$$= \{x \in X \mid f(x) > \tfrac{a}{c}\}$$
$$= f^{-1}((\tfrac{a}{c}, \infty))$$

for $c < 0$ $\quad (c \cdot f)^{-1}((a,\infty)) = f^{-1}((-\infty, \tfrac{a}{c}))$.

②. $$(f+g)^{-1}((a,\infty)) = \{x \in X \mid f(x) + g(x) > a\}.$$
$$= \{x \in X \mid f(x) > a - g(x)\}$$
$$= \bigcup \{x \in X \mid f(x) > r > a - g(x)\}.$$

$\{ x \in X \mid f(x) > a - g(x) \}$

$$= \bigcup_{r \in \mathbb{Q}} \{ x \in X \mid f(x) > r > a - g(x) \}.$$

③ $f \cdot g = \frac{1}{2}\left(f^2 + g^2 - (f-g)^2\right).$

$\square$

Limits of measurable functions

Consider the extended real line: $\overline{\mathbb{R}} := [-\infty, \infty]$.

Define $\mathcal{B}_{\overline{\mathbb{R}}} := \{ E \subset \overline{\mathbb{R}} \mid E \cap \mathbb{R} \in \mathcal{B}_{\mathbb{R}} \}$

Convention: $0 \cdot \infty = 0$

Prop: $(X, \mathcal{M})$ measurable space, $f, g : X \to \overline{\mathbb{R}}$ are measurable $c \in \mathbb{R}$. Then $c \cdot f$, $f + g$, $f \cdot g$ are measurable.

Prop: $(X, \mathcal{M})$ measurable space, $f_j : X \to \overline{\mathbb{R}}$ measurable for all $j \in \mathbb{N}$. then

① $g_1(x) := \sup_j f_j(x)$

② $g_2(x) := \inf_j f_j(x)$

③ $g_3(x) := \limsup_{j \to \infty} f_j(x)$

④ $g_4(x) := \liminf_{j \to \infty} f_j(x)$

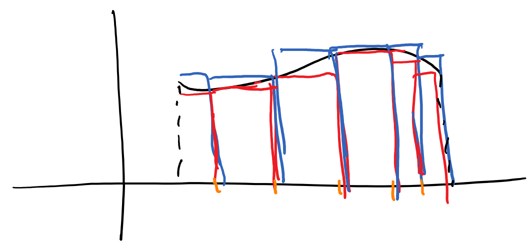
are all measurable functions from $X \to \overline{\mathbb{R}}$.

If $f(x) := \lim_{j \to \infty} f_j(x)$ exists $\forall x \in X$ then f is measurable.

Integration / Simple Functions

Recall: Riemann-Stieltjes Integration

Idea: Use polygonal approximations to estimate the integral of a function $f:[a,b]\to\mathbb{R}$



Caveat: f needs to be mostly continuous.

In order to extend to measurable functions, we will need a different type of approximation