1) Introduction

a.) Discussion of syllabus

(Note: If conditions change we will make appropriate)

2.) What are we studying?

Suppose one would like to solve the linear partial differential equation,

$$i \frac{d}{dt} u(t,x) = \left(\frac{d}{dx}\right)^2 u(t,x).$$

where $t \in \mathbb{R}$, $x \in \mathbb{R}$, $u: \mathbb{R}\times\mathbb{R} \to \mathbb{C}$, $u(0,x)$ is given.

u is 2π periodic in the x variable.

How do we solve this equation?

Suppose that for each $t \in \mathbb{R}$, $u(t,x)$ is a finite linear combination of eigenfunctions of $\left(\frac{d}{dx}\right)^2$

$$u(t,x) = \sum_{n=1}^{N} u_n(t)\, e^{inx}$$

Now we have a system of linear ODEs!

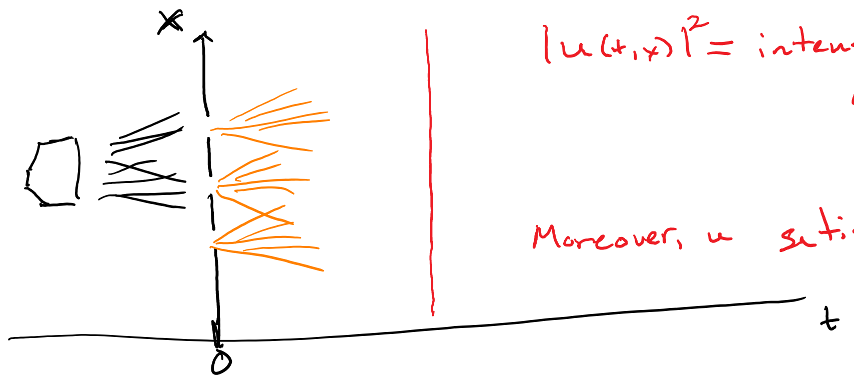
$$\sum_{n=1}^{N} i \frac{d}{dt} u_n(t)\, e^{inx} = \sum_{n=1}^{N} \left(\frac{d}{dx}\right)^2 u_n(t)\, e^{inx}$$

$$\sum_{n=1}^{N} i \frac{d}{dt} u_n(t)\, e^{inx} = \sum_{n=1}^{N} -n^2 u_n(t)\, e^{inx}$$

$\Longleftrightarrow \quad i\frac{d}{dt} u_n(t) = -n^2 u_n(t) \quad \Rightarrow \quad \boxed{u_n(t) = e^{in^2 t} u_n(0).}$

and $\quad u(t,x) = \sum_{n=1}^{N} u_n(0) e^{in^2 t} e^{inx} \quad$ for all $t \in \mathbb{R}$.

What happens when $u(t,x)$ is not so nice? For example consider light shining through a grating



$|u(t,x)|^2$ = intensity of light at distance, t, and height, x.

Moreover, u satisfies $\quad i\frac{d}{dt} u = \left(\frac{d}{dx}\right)^2 u$

Then $\quad u(0,x) = a\chi_{I_1}(x) + a\chi_{I_2}(x) + a\chi_{I_3}(x) \quad$ which can not be written as a finite linear combination of eigenfunctions.

However, we may approximate $u(0,x)$ by longer and longer finite linear combinations of eigenfunctions:

$$u_N(t,x) \xrightarrow{N\to\infty} u(t,x)$$

Many questions arise, among them:

a.) In what sense does $u_N \to u$?

- Uniformly? Not likely
- Pointwise? Maybe, but with exceptions.
  - How do we quantify / qualify exceptions

b.) If u_N solves the PDE for each N, b.) If u_N solves the PDE for each N, does u solve the PDE and in what sense?

We start here.

Topological Preliminaries

Metric Spaces

Definition: A metric space (X, ρ) is a non-empty set, X, with a function

$$\rho : X \times X \to [0, \infty)$$

satisfying

i.) $\rho(x,y) = 0 \iff x = y$

ii.) $\rho(x,y) = \rho(y,x)$

iii.) (Δ ineq) $\rho(x,z) \leq \rho(x,y) + \rho(y,z)$.

Examples

- Euclidean Space, $(\mathbb{R}^d, \|\cdot - \cdot\|)$ $\quad x = (x_1, \dots, x_d)$.

$$\rho(x,y) := \|x-y\| = \left(\sum_{i=1}^{d} (x_i - y_i)^2 \right)^{1/2}$$

- Discrete Metric (X, ρ)

where $$\rho(x,y) := \begin{cases} 0 & \text{if } x=y \\ 1 & \text{if } x \neq y \end{cases}$$

- $$\mathcal{C}([0,1]) = \left(\{ \text{continuous functions}, f:[0,1] \to \mathbb{R} \}, \|\cdot\cdot\cdot\|_\infty \right).$$

$$\rho(f,g) := \|f-g\|_\infty = \sup_{x \in [0,1]} |f(x)-g(x)|$$

- $$\ell^2(\mathbb{N}) := \left(\{ \underline{x} = (x_n)_{n=1}^\infty : x_n \in \mathbb{R} \}, \|\cdot\cdot\cdot\|_{\ell^2} \right)$$

where $$\rho(\underline{x},\underline{y}) = \|\underline{x}-\underline{y}\|_{\ell^2} = \left(\sum_{n=1}^\infty |x_n - y_n|^2 \right)^{1/2}.$$

Open and Closed Sets in a Metric Space (X,ρ)

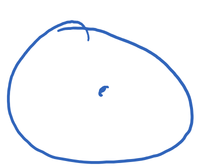
We denote an open ball in (X,ρ), with center $x \in X$ and radius, $r>0$ by

$$B(x,r) := \{ y \in X : \rho(x,y) < r \}.$$

Examples

$(\mathbb{R}^d, \|.\|)$

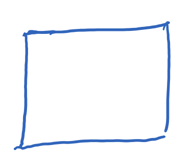
$B(x,r)$



$(\mathbb{R}^d, \|\cdot\|_\infty)$

where $\|x\|_\infty := \max_{i=1,\dots,d} |x_i|$

$B(x,r)$



Definitions

1) $G \subset X$ is open if for all $x \in G$, $\exists r > 0$ such that $B(x,r) \subset G$

Note: The symbol, $\subset$, represents non proper subsets

2.) $F \subset X$ is closed if F^c is open.

Note: The open ball is an open set.

Proposition

1.) X, ϕ are open sets
2.) The union of any collections of open sets is open
3.) The intersection of a finite collection of open sets is open.

Definitions

Let $E \subset X$

① Interior of E

$$\text{int } E = \bigcup_{\substack{G \subset E \\ G \text{ open}}} G$$

, int E = "largest" open set contained in E
with respect to inclusion

② Closure of E

$$\overline{E} = \bigcap_{\substack{F \supset E \\ F \text{ closed}}}$$

, $\overline{E}$ = "smallest" closed set containing E.