Homework Problems
Math 524, Autumn 2021
Due 11:00 pm, December 10, 2021

Instructions: Please write your solution to each problem on a separate page, and please include the full problem statement at the top of the page. All solutions must be written in legible handwriting or typed (in each case, the text should be of a reasonable size).

Your solutions to all problems should be written in complete sentences, with proper grammatical structure.

If your solutions are not typed, you must scan your written solutions and submit the digital copy. When submitting problems through LaTex, the LaTex source file (.tex) must be included in the submission.

1. * (Folland, Chapter 3, Section 4, Problem 22) If $f \in L^1\left(\mathbb{R}^d\right), f \neq 0$, there exist $C, R > 0$ such that $Hf(x) \geq C|x|^{-d}$ for $|x| > R$. Hence $m(\{x : Hf(x) > \alpha\}) \geq C'/\alpha$ when α is small, so the estimate in the maximal theorem is essentially sharp.

2. * (Prelim 2016) Suppose that a (X, ρ) is a compact metric space and $f : X \to \mathbb{R}$ is a function. If for every $t \in \mathbb{R}$, $f^{-1}([t, \infty))$ is closed, then there is some $x_0 \in X$ so that $f(x_0) = \sup_{x \in X} f(x) < \infty$.

3. * Show that for any open set $U \subset \mathbb{R}^d$, and $\delta > 0$, there exists a countable collection $\mathcal{G}$ of disjoint closed balls in U such that diam B $\leq \delta$ for all $B \in \mathcal{G}$, and

$$m\left(U \backslash \bigcup_{B \in \mathcal{G}} B\right) = 0$$

4. * Let μ be a doubling measure (i.e. $\exists C \geq 1$ such that $\mu(B(x, 2r)) \leq C\mu(B(x, r))\ \forall x \in \mathbb{R}^d, \forall r > 0$) on $\mathbb{R}^d$. Assume that $\mu(B(x_0, 1)) < \infty$ for some $x_0 \in \mathbb{R}^d$

   (a) Show that if K is compact then $\mu(K) < \infty$.

   (b) Prove that if $E \in \mathcal{B}_{\mathbb{R}^n}$ with $\mu(E) < \infty$ then given $\epsilon > 0$ there exists a compact set K with $K \subset E$ and $\mu(E \backslash K) < \epsilon$.

5. * Let $\mathcal{I} = \{I_k\}_{k=1}^N$ be a finite collection of bounded, closed intervals in $\mathbb{R}$. Show that there exists a subcollection of $\mathcal{I}$, $\{I_{k_\ell}\} \subset \mathcal{I}$, such that

$$\bigcup_k I_k \subset \bigcup_\ell I_{k_\ell}$$

   and for every $x \in \bigcup_k I_k$, there at most two intervals in $\{I_{k_\ell}\}$ to which x belongs.

6. (Folland, Chapter 3, Section 4, Problem 22) If E is a Borel set in $\mathbb{R}^n$, the density $D_E(x)$ of E at x is defined as

$$D_E(x) = \lim_{r \to 0} \frac{m(E \cap B(r, x))}{m(B(r, x))}$$

   whenever the limit exists.

   (a) Show that $D_E(x) = 1$ for a.e. $x \in E$ and $D_E(x) = 0$ for a.e. $x \in E^c$.

   (b) Find examples of E and x such that $D_E(x)$ is a given number $\alpha \in (0, 1)$, or such that $D_E(x)$ does not exist.

7. (Folland, Chapter 3, Section 5, Problem 30) Construct an increasing function on $\mathbb{R}$ whose set of discontinuities is $\mathbb{Q}$.

8. (Folland, Chapter 3, Section 5, Problem 33) If F is increasing on $\mathbb{R}$, then $F(b) - F(a) \geq \int_a^b F'(t)dt$.

9. * (Folland, Chapter 3, Section 5, Problem 37) Suppose $F : \mathbb{R} \to \mathbb{C}$. There is a constant M such that $|F(x) - F(y)| \leq M|x - y|$ for all $x, y \in \mathbb{R}$ (that is, F is **Lipschitz continuous**) iff F is absolutely continuous and $|F'| \leq M$ m-a.e.

10. (Folland, Chapter 3, Section 5, Problem 41) Let $A \subset [0, 1]$ be a Borel set such that $0 < m(A \cap I) < m(I)$ for every subinterval I of $[0, 1]$ (Exercise 33 , Chapter 1).

    (a) Let $F(x) = m([0, x] \cap A)$. Then F is absolutely continuous and strictly increasing on $[0, 1]$, but $F' = 0$ on a set of positive measure.

    (b) Let $G(x) = m([0, x] \cap A) - m([0, x] \setminus A)$. Then G is absolutely continuous on $[0, 1]$, but G is not monotone on any subinterval of $[0, 1]$.