Homework Problems
Math 524, Autumn 2021
Due 11:00 pm, November 29, 2021

Instructions: Please write your solution to each problem on a separate page, and please include the full problem statement at the top of the page. All solutions must be written in legible handwriting or typed (in each case, the text should be of a reasonable size).

Your solutions to all problems should be written in complete sentences, with proper grammatical structure.

If your solutions are not typed, you must scan your written solutions and submit the digital copy. When submitting problems through LaTex, the LaTex source file (.tex) must be included in the submission.

Definition 0.0.1. *Let $d \in \mathbb{Z}_+$, $\delta > 0$, and $s \in [0, \infty)$. For $E \subset \mathbb{R}^d$ define*

$$\mathcal{H}^s_\delta(E) := \inf \left\{ \sum_{i=1}^{\infty} (diam(F_i))^s \ : \ E \subset \bigcup_{i=1}^{\infty} F_i,\ diam(F_i) < \delta \right\}$$

and the s-Hausdorff measure of E as

$$\mathcal{H}^s(E) := \sup_{\delta > 0} \mathcal{H}^s_\delta(E)$$

Definition 0.0.2. *Let $d \in \mathbb{Z}_+$ and let $E \subset \mathbb{R}^d$ be a Borel set. Define the Hausdorff dimension of E by*

$$\dim_{\mathcal{H}}(E) := \inf\{s \in [0, \infty) : \mathcal{H}^s(E) = 0\} = \sup\{s \in [0, \infty) : \mathcal{H}^s(E) = \infty\}$$

1. * Show that $\mathcal{H}^s$ is a metric outer measure.

2. * Let $\mathcal{C}$ be the middle-third Cantor set. Show that $\dim_{\mathcal{H}}(\mathcal{C}) \leq \log(2)/\log(3)$.

3. Let $f : \mathbb{R}^d \to \mathbb{R}^k$ be a linear function. Let $E \subset \mathbb{R}^d$ be a Borel set. Show that $\dim_{\mathcal{H}}(f(E)) \leq \dim_{\mathcal{H}}(E)$. In particular, show that for every $s \in [0, \infty)$ there is a constant $C_s > 0$ such that $\mathcal{H}^s(f(E)) \leq C_s \mathcal{H}^s(E)$.

4. Suppose F is a closed set in $\mathbb{R}$, whose complement has finite measure, and let $\delta(x)$ denote the distance from x to F, that is

$$\delta(x) = d(x, F) = \inf\{|x - y| : y \in F\}$$

   Consider

$$I(x) = \int_{\mathbb{R}} \frac{\delta(y)}{|x - y|^2}\, dm(y)$$

   (a) Prove that δ is continuous, by showing that it satisfies the Lipschitz condition

$$|\delta(x) - \delta(y)| \leq |x - y|$$

   (b) Show that $I(x) = \infty$ for each $x \notin F$.

   (c) Show that $I(x) < \infty$ for *a. e.* $x \in F$. This may be surprising in view of the fact that the Lipschitz condition cancels only one power of $|x - y|$ in the integrand of I. [Hint: Investigate $\int_F I(x)dx$.]

5. * (Folland, Chapter 2, Section 4, Problem 34) Suppose $|f_n| \leq g \in L^1(\mu)$ and $f_n \to f$ in measure.

   (a) $\int f = \lim \int f_n$.

   (b) $f_n \to f$ in $L^1(\mu)$.

6. (Folland, Chapter 2, Section 4, Problem 42) Let μ be counting measure on $\mathbb{N}$. Then $f_n \to f$ in measure iff $f_n \to f$ uniformly.

7. * (Folland, Chapter 2, Section 5, Problem 48) Let $X = Y = \mathbb{N}$, $\mathcal{M} = \mathcal{N} = \mathcal{P}(\mathbb{N})$, $\mu = \nu =$ counting measure. Define $f(m,n) = 1$ if $m = n$, $f(m,n) = -1$ if $m = n+1$, and $f(m,n) = 0$ otherwise. Then $\int |f| d(\mu \times \nu) = \infty$, and $\iint f d\mu d\nu$ and $\iint f d\nu d\mu$ exist and are $\iint f d\mu d\nu \neq \iint f d\nu d\mu$.

8. Let f be a Lebesgue measurable finite-valued function on $[0,1]$, and supposed that $h(x,y) = |f(x) - f(y)|$ is integrable in $[0,1] \times [0,1]$. Show that $f(x)$ is integrable in $[0,1]$.

9. * Suppose f is integrable on $\mathbb{R}^d$. For each $\alpha > 0$, let $E_\alpha = \{x : |f(x)| > \alpha\}$. Prove that

$$\int_{\mathbb{R}^d} |f(x)| \, dm(x) = \int_0^\infty m\left(E_\alpha\right) \, d\alpha$$

10. * Suppose f are g are Lebesgue measurable functions on $\mathbb{R}^d$.

    (a) Prove that $f(x-y)g(y)$ is measurable in $\mathbb{R}^{2d}$.

    (b) If f are g are integrable on $\mathbb{R}^d$, then $f(x-y)g(y)$ is integrable in $\mathbb{R}^{2d}$.

    (c) Recall the definition of the convolution of f and g given by

    $$(f * g)(x) = \int_{\mathbb{R}^d} f(x-y)g(y) \, dm(y)$$

    Show that $f * g$ is well defined for m-a.e. x (that is $f(x-y)g(y)$ is integrable on $\mathbb{R}^d$ for m-a.e. x).

    (d) Show that $f * g$ is integrable whenever f and g are integrable, and that

    $$\|f * g\|_{L^1(\mathbb{R}^d)} \leq \|f\|_{L^1(\mathbb{R}^d)} \|g\|_{L^1(\mathbb{R}^d)}$$