Homework Problems
Math 524, Autumn 2021
Due 11:00 pm, November 11, 2021

Instructions: Please write your solution to each problem on a separate page, and please include the full problem statement at the top of the page. All solutions must be written in legible handwriting or typed (in each case, the text should be of a reasonable size).

Your solutions to all problems should be written in complete sentences, with proper grammatical structure.

If your solutions are not typed, you must scan your written solutions and submit the digital copy. When submitting problems through LaTex, the LaTex source file (.tex) must be included in the submission.

1. * (Folland, Chapter 2, Section 2, Problem 12) Prove tha following proposition:

   Proposition 0.0.1. *If $f \in L^1(\mu)$, then $\mu(\{x \in X : f(x) = \infty\}) = 0$ and $\{x \in X : f(x) > 0\}$ is σ-finite.*

2. * (Folland, Chapter 2, Section 2, Problem 14) For $f \in L^+$, define the function, $\lambda(E) := \int_E f \, d\mu$, for all $E \in \mathcal{M}$. Then λ is a measure on $\mathcal{M}$, and for any $g \in L^+$, $\int g \, d\lambda = \int fg \, d\mu$.

3. (Folland, Chapter 2, Section 2, Problem 15) Show without using dominated convergence theorem: If $\{f_n\} \subset L^+$, f_n decreases pointwise to f, and $\int f_1 \, d\mu < \infty$, then $\int f \, d\mu = \lim_{n\to\infty} \int f_n \, d\mu$.

4. * (Folland, Chapter 2, Section 2, Problem 16) If $f \in L^+$ and $\int f \, d\mu < \infty$, for every $\varepsilon > 0$ there exists $E \in \mathcal{M}$ such that $\mu(E) < \infty$ and $\int_E f \, d\mu > \int_X f \, d\mu - \varepsilon$.

5. (Folland, Chapter 2, Section 3, Problem 19) Suppose $\{f_n\}_{n=1}^\infty \subset L^1(\mu)$ and $f_n \to f$ uniformly.

   (a) If $\mu(X) < \infty$, then $f \in L^1(\mu)$ and $\int f_n \, d\mu \to \int f \, d\mu$.

   (b) If $\mu(X) = \infty$, then the conclusions of (a) can fail.

6. * Consider a complete, finite measure space $(X, \mathcal{M}, \mu)$. Let $\{f_n\}_{n=1}^\infty \subset L^1(\mu)$ be a sequence such that $\sup_n \int |f_n| \, d\mu < \infty$. Furthermore, assume that for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\int_E |f_n| \, d\mu < \varepsilon$ whenever $\mu(E) < \delta$. Show that

   - $\sup_n \int_{\{|f_n| \geq M\}} |f_n| \, d\mu \to 0$ as $M \to \infty$.
   - $\sup_n \int_{\{|f_n| \leq \delta\}} |f_n| \, d\mu \to 0$ as $\delta \to 0$.

7. (Folland, Chapter 2, Section 3, Problem 20) If $f_n, g_n, f, g \in L^1(\mu)$, $f_n \to f$ and $g_n \to g$ μ-a.e., $|f_n| \leq g_n$, and $\int g_n \, d\mu \to \int g \, d\mu$, then $f_n \, d\mu \to \int f \, d\mu$.

8. * (Folland, Chapter 2, Section 3, Problem 26) If $f \in L^1(m)$ and $F(x) = \int_{-\infty}^x f(t) \, dt$, then F is continuous on $\mathbb{R}$.

9. (Folland, Chapter 2, Section 4, Problem 36) If $\mu(E_n) < \infty$ for $n \in \mathbb{N}$ and and $\chi_{E_n} \to f$ in L^1, then f is the indicator function of a measurable set.

10. * (Folland, Chapter 2, Section 4, Problem 44) If $f : [a, b] \to \mathbb{C}$ is Lebesgue measurable and $\varepsilon > 0$, there is a compact set $E \subset [a, b]$ such that $\mu(E^c) < \varepsilon$ and $f|_E$ is continuous.