Homework Problems
Math 524, Autumn 2021
Due 11:00 pm, October 28, 2021

Instructions: Please write your solution to each problem on a separate page, and please include the full problem statement at the top of the page. All solutions must be written in legible handwriting or typed (in each case, the text should be of a reasonable size).

Your solutions to all problems should be written in complete sentences, with proper grammatical structure.

If your solutions are not typed, you must scan your written solutions and submit the digital copy. When submitting problems through LaTex, the LaTex source file (.tex) must be included in the submission.

1. * Prove the following theorem:

   Theorem 1. *Every open set in* $\mathbb{R}^d$ *can be written as a countable union of dyadic cubes.*

2. Given a collection, $\{\mathcal{M}_\alpha\}_\alpha$, of σ-algebras, show that $\bigcap_\alpha \mathcal{M}_\alpha$ is a σ-algebra.

3. * (Folland, Chapter 1, Section 3, Problem 9) If $(X, \mathcal{M}, \mu)$ is a measure space and $E, F \in \mathcal{M}$, then

$$\mu(E) + \mu(F) = \mu(E \cup F) + \mu(E \cap F).$$

4. (Folland, Chapter 1, Section 5, Problem 29) Let E be a Lebesgue measurable set.

   (a) If $E \subset N$ where N is the nonmeasurable set described in Section 1.1, then $m(E) = 0$

   (b) If $m(E) > 0$, then E contains a nonmeasurable set.

5. * (Folland, Chapter 1, Section 5, Problem 30) If $E \subset \mathbb{R}$ is a Lebesgue measurable subset and $m(E) > 0$, then for any $\alpha < 1$ there is an open interval I such that $m(E \cap I) > \alpha m(I)$.

6. * Suppose $E \subset \mathbb{R}^d$ and $\mathcal{O}_n$ is the open set

$$\mathcal{O}_n = \left\{ x \in \mathbb{R}^d : d(x, E) < \frac{1}{n} \right\}$$

   Show:

   (a) If E is compact, then $m(E) = \lim_{n\to\infty} m\left(\mathcal{O}_n\right)$.

   (b) However, this may be false for E closed and unbounded; or E open and bounded. Provide examples.

7. Suppose $\{E_k\}_{k=1}^{\infty}$ is a countable family of measurable subsets of $\mathbb{R}^d$ and that

$$\sum_{k=1}^{\infty} m\left(E_k\right) < \infty$$

   Let

$$E = \limsup_{k\to\infty} \left(E_k\right) = \left\{ x \in \mathbb{R}^d : x \in E_k \text{ for infinitely many } k \right\}$$

   (a) Show that E is (Lebesgue) measurable.

   (b) Show that $m(E) = 0$.

   Hint: Write $E = \cap_{n=1}^{\infty} \cup_{k\geq n} E_k$.

8. (Prelim 2007) Let X be a compact metric space, and μ be a finite positive Borel measure on X. Suppose that $\mu(\{x\}) = 0$ for every $x \in X$. Prove that for every $\varepsilon > 0$ there is a $\delta > 0$ such that, if E is any Borel subset of X having diameter less than δ, then $\mu(E) < \varepsilon$

9. (Folland, Chapter 2, Section 1, Problem 2) Suppose $f, g : X \to \overline{\mathbb{R}}$ are measurable.

   (a) fg is measurable (Note: Let $0 \cdot \pm\infty := 0$)

   (b) Fix $a \in \overline{\mathbb{R}}$ and define

$$h(x) = \begin{cases} a & f(x) = -g(x) = \pm\infty \\ f(x) + g(x) & \text{otherwise.} \end{cases}$$

   Show that h is measurable.

10. * (Folland, Chapter 2, Section 1, Problem 8) If $f : \mathbb{R} \to \mathbb{R}$ is monotone, then f is Borel measurable.

11. (Folland, Chapter 2, Section 1, Problem 9) Let C denote the middle-thirds Cantor set. Let $x \in [0, 1]$ have the ternary expansion $(a_n)_{n=1}^{\infty}$ (i.e. $x = \sum_{n=1}^{\infty} \frac{a_n}{3^n}$). Define the function, $N : [0, 1] \setminus C \to \mathbb{Z}_+$, where

$$N(x) := \inf \{n \in \mathbb{Z}_+ \ : \ a_n = 1\}.$$

   Finally, define the function $f : [0, 1] \to [0, 1]$ by

$$f(x) := \begin{cases} \frac{1}{2^{N(x)}} + \sum_{n=1}^{N(x)-1} \frac{a_n}{2^{n+1}} & \text{for } x \in [0, 1] \setminus C \\ \sum_{n=1}^{\infty} \frac{a_n}{2^{n+1}} & \text{for } x \in C. \end{cases}$$

   and the function $g : [0, 1] \to [0, 2]$ by $g(x) := f(x) + x$.

   (a) Show that g is a bijection from $[0, 1]$ to $[0, 2]$, and $h := g^{-1}$ is continuous from $[0, 2]$ to $[0, 1]$

   (b) Show that $m(g(C)) = 1$.

   (c) Show that $g(C)$ contains a Lebesgue nonmeasurable set, A. Show that $B := g^{-1}(A)$ is Lebesgue measurable but not Borel.

   (d) There exist a Lebesgue measurable function, F, and a continuous function, G, on $\mathbb{R}$ such that $F \circ G$ is not Lebesgue measurable.

12. * (Folland, Chapter 2, Section 1, Problem 10) Let $(X, \mathcal{M}, \mu)$ be a measure space. Prove the following proposition:

   Proposition 2. *The following implications are valid iff the measure, μ, is complete:*

   (a) If $f : X \to \overline{\mathbb{R}}$ is measurable and $f = g$ μ-almost everywhere, then g is measurable

   (b) If $f_n : X \to \overline{\mathbb{R}}$ is measurable for all $n \in \mathbb{N}$ and $f_n \to f$ μ-almost everywhere, then f is measurable.

13. Let $(X, \mathcal{M})$ be a measurable space. Let $f : X \to \mathbb{R}^n, f = (f_1, \cdots, f_n)$. Show that f is $(\mathcal{M}, \mathcal{B}_{\mathbb{R}^n})$-measurable if and only if for each $i = 1, \cdots, n, f_i : X \to \mathbb{R}$ is $(\mathcal{M}, \mathcal{B}_{\mathbb{R}})$-measurable.