Homework Problems
Math 524, Autumn 2021
Due 11:00 pm, October 14, 2021

Instructions: Please write your solution to each problem on a separate page, and please include the full problem statement at the top of the page. All solutions must be written in legible handwriting or typed (in each case, the text should be of a reasonable size).

Your solutions to all problems should be written in complete sentences, with proper grammatical structure.

If your solutions are not typed, you must scan your written solutions and submit the digital copy. When submitting problems through LaTex, the LaTex source file (.tex) must be included in the submission.

1. Suppose $f : \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying

   (a) $0 \leq f(t) \leq 1$ for all $t \in \mathbb{R}$,

   (b) $f(t+2) = f(t)$ for all $t \in \mathbb{R}$,

   (c)
$$f(t) = \begin{cases} 0, & t \in [0, \frac{1}{3}] \\ 1, & t \in [\frac{2}{3}, 1] \end{cases}$$

   Define the function $\Phi : [0,1] \to \mathbb{R}^2$ by $\Phi(t) := (x(t), y(t))$, where
$$x(t) = \sum_{n=1}^{\infty} 2^{-n} f(3^{2n-1}t), \quad y(t) = \sum_{n=1}^{\infty} 2^{-n} f(3^{2n}t).$$

   Prove that Φ is continuous and that Φ maps $[0,1]$ onto $[0,1] \times [0,1] \subset \mathbb{R}^2$. In fact, show that Φ maps the middle-thirds Cantor set onto $[0,1] \times [0,1]$.

2. Let $\{x_n\}_n \subset \mathbb{R}$, and $x \in \mathbb{R}$. Show that $x = \lim_{n\to\infty} x_n$ if and only if every subsequence of $\{x_n\}_{n\geq 1}$ has in turn a subsequence which converges to x.

3. Prove that $E \subset \mathbb{R}^n$ is compact if and only if E is closed and bounded.

4. * Let $F \subset V \subsetneq \mathbb{R}$ where F is closed and V is open. Prove that there exists an open set $U \subset \mathbb{R}$ satisfying
$$F \subset U \subset \overline{U} \subset V.$$

5. Let (X, ρ) be a metric space. For a closed set $E \subset X$, define $\operatorname{diam} E := \sup\{\rho(x,y) : x, y \in E\}$. Prove that X is complete if and only if for any decreasing sequence $\{A_n\}$ of non-empty closed subsets of X (i.e. $A_1 \supset A_2 \cdots \supset A_{n-1} \supset A_n \cdots$, $A_i \neq \emptyset, A_i$ closed) such that
$$\lim_{n\to\infty} \operatorname{diam} A_n = 0$$
   then
$$\cap_{n=1}^{\infty} A_n = \{x\} \quad \text{for some} \quad x \in X.$$

6. Let $g : \mathbb{R} \to [0,1]$ be a non-decreasing function. Prove that the set of discontinuities of g is a countable.

7. * Suppose that $f_n : \mathbb{R} \to [0,1]$ for $n = 1, 2, \ldots$, and that each one of these functions is non-decreasing, that is, $f_n(x) \leq f_n(y)$ if $x \leq y$, for all n, x and y. Prove that there exist a function $g : \mathbb{R} \to [0,1]$, a countable set $A \subset \mathbb{R}$, and a subsequence f_{n_k} such that $\lim_{k\to\infty} f_{n_k}(x) = g(x)$ for all $x \in \mathbb{R} \backslash A$

8. * Properties of the exterior measure m_* in $\mathbb{R}^d$. Recall for $E \subset \mathbb{R}^d$

$$m_*(E) := \inf\left\{\sum_j |Q_j| : E \subset \cup_j Q_j : Q_j \text{ closed cubes }\right\}.$$

For a cube $Q \subset \mathbb{R}^d$, $|Q|$ denotes the volume of Q.

(a) Prove that the exterior (outer) measure m_* of a closed cube in $\mathbb{R}^d$ is its volume.

(b) Prove that if $E \subset \mathbb{R}^d$ is a countable union of closed cubes such that their interiors are disjoint, that is

$$E = \cup_{i=1}^{\infty} Q_i \text{ with } \operatorname{int} Q_i \cap \operatorname{int} Q_j = \emptyset$$

then

$$m_*(E) = \sum_{j=1}^{\infty} m_*\left(Q_j\right).$$

9. * Prove that if $E \subset \mathbb{R}^d$ and $E = E_1 \cup E_2$, with $d\left(E_1, E_2\right) > 0$ then

$$m_*(E) = m_*\left(E_1\right) + m_*\left(E_2\right)$$

10. Let $s, \delta \in (0,1)$. Consider the outer measure on $\mathbb{R}$, μ_δ^s, defined by

$$\mu_\delta^s(E) := \inf\left\{\sum_j |I_j|^s : E \subset \cup_j I_j : I_j \text{ closed intervals }, |I_j| \leq \delta\right\}.$$

For an interval $I \subset \mathbb{R}$, $|I|$ denotes the length of I. Prove that if E is an interval and $\delta < |E|$, then

$$\mu_\delta^s(E) \geq \delta^{s-1}|E| - \delta^s$$

11. Let F be a subset of $[0,1]$ constructed in the same manner as the Cantor ternary set except that each of the intervals removed at the nth step has length $\alpha 3^{-n}$ with $0 < \alpha < 1$. Then F is closed, F^c is dense in $[0,1]$ and $m_* F = 1 - \alpha$. Such set F is called a generalized Cantor set.

12. * Let (X, ρ) be a non-empty complete metric space. Let $f : X \to X$ be such that there exists $\lambda \in (0,1)$ satisfying

$$\rho(f(x), f(y)) \leq \lambda \rho(x, y) \text{ for all } x, y \in X$$

Prove that there exists a unique point $u \in X$ such that $f(u) = u$.

13. (Prelim 2001) Let (X, d) be a metric space and $K \subset X$ be compact. Let $\{G_i\}_{i \in I}$ be an open cover of K. Show that there exits $\epsilon > 0$ such that every subset S of K of diameter $< \epsilon$ is contained in one of the sets G_i (i.e. that there exists $i = i(S)$ such that $S \subset G_i$)

14. * Prove the following statement:

(X, ρ) is compact if and only if every collection $\mathcal{F}$ of closed sets with the finite intersection property has non-empty intersection.