Reduced words and a formula of Macdonald

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Based on joint work with:
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Outline

Permutations and Reduced Words

Macdonald's Reduced Word Formula

Generalizations of Macdonald's Formula

Open Problems

Permutations

Permutations are fundamental objects in mathematics, computer science, game theory, economics, physics, chemistry and biology.

Notation.

- S_n is the symmetric group of permutations on n letters.
- $w \in S_n$ is a bijection from $[n] := \{1, 2, \ldots, n\}$ to itself denoted in *one-line notation* as $w = [w(1), w(2), \ldots, w(n)]$.
- $s_i = (i \leftrightarrow i+1) =$ adjacent transposition for $1 \leq i < n$.

Example. $w = [3, 4, 1, 2, 5] \in S_5$ and $s_4 = [1, 2, 3, 5, 4] \in S_5$.

$ws_4 = [3, 4, 1, 5, 2]$ and $s_4 w = [3, 5, 1, 2, 4]$.

Permutations

Presentation of the Symmetric Group.

Fact. S_n is generated by $s_1, s_2, \ldots, s_{n-1}$ with relations

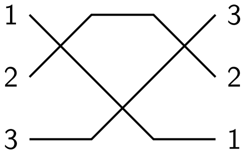
$$\begin{aligned} &s_i s_i = 1 \\ &(s_i s_j)^2 = 1 \text{ if } |i-j| > 1 \\ &(s_i s_{i+1})^3 = 1 \end{aligned}$$

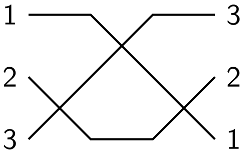
For each $w \in S_n$, there is some expression $w = s_{a_1} s_{a_2} \cdots s_{a_p}$.
If p is minimal, then

- $\ell(w) =$ *length of w* $=$ p,
- $s_{a_1} s_{a_2} \cdots s_{a_p}$ is a *reduced expression* for w,
- $a_1 a_2 \ldots a_p$ is a *reduced word* for w.

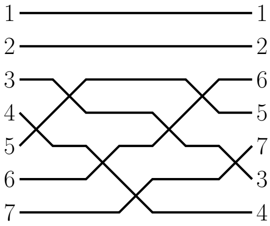
Reduced Words and Reduced Wiring Diagrams

Example. 121 and 212 are reduced words for $[3, 2, 1]$.





Example. 4356435 is a reduced word for $[1, 2, 6, 5, 7, 3, 4] \in S_7$.



Reduced Words

Key Notation. $R(w)$ is the set of all reduced words for w.

Example. $R([3,2,1]) = \{121, 212\}$.

Example. $R([4,3,2,1])$ has 16 elements:

321323	323123	232123	213213
231213	321232	132132	312132
132312	312312	123212	213231
231231	212321	121321	123121

Example. $R([5,4,3,2,1])$ has 768 elements.

Counting Reduced Words

Question. How many reduced words are there for w?

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Theorem.(Stanley, 1984) For $w_0^n := [n, n-1, \ldots, 2, 1] \in S_n$,

$$|R(w_0^n)| = \frac{\binom{n}{2}!}{1^{n-1}\ 3^{n-2}\ 5^{n-3}\ \cdots (2n-3)^1}.$$

Observation: The right side is equal to the number of standard Young tableaux of staircase shape $(n-1, n-2, \ldots, 1)$.

Counting Standard Young Tableaux

Defn. A *partition* of a number n is a weakly decreasing sequence of positive integers

$$\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0)$$

such that $n = \sum \lambda_i = |\lambda|$.

Partitions can be visualized by their *Ferrers diagram*

$$(6,5,2) \longrightarrow \begin{array}{|c|c|c|c|c|c|} \hline & & & & & \\ \hline & & & & & \multicolumn{1}{c}{} \\ \cline{1-5} & & \multicolumn{4}{c}{} \\ \cline{1-2} \end{array}$$

Def. A standard Young tableau T of shape λ is a bijective filling of the boxes by $1, 2, \ldots, n$ with rows and columns increasing.

Example. $T = \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & 3 & 6 & 8 \\ \hline 4 & 5 & 9 & \multicolumn{2}{c}{} \\ \cline{1-3} 7 & \multicolumn{4}{c}{} \\ \cline{1-1} \end{array}$

The standard Young tableaux (SYT) index the bases of S_n-irreps.

Counting Standard Young Tableaux

Hook Length Formula.(Frame-Robinson-Thrall, 1954) If λ is a partition of n, then

$$\#SYT(\lambda) = \frac{n!}{\prod_{c \in \lambda} h_c}$$

where h_c is the *hook length* of the cell c, i.e. the number of cells directly to the right of c or below c, including c.

Example. Hook lengths of $\lambda = (5, 3, 1)$:

7	5	4	2	1
4	2	1		
1				

So, $\#SYT(5, 3, 1) = \frac{8!}{7 \cdot 5 \cdot 4 \cdot 3 \cdot 4 \cdot 2} = 162$.

Remark. Notable other proofs by Greene-Nijenhuis-Wilf '79 (probabalistic), Krattenthaler '95 (bijective), Novelli-Pak-Stoyanovskii '97 (bijective), Bandlow '08.

Counting Reduced Words

Theorem.(Edelman-Greene, 1987) For all $w \in S_n$,

$$|R(w)| = \sum a_{\lambda,w} \# SYT(\lambda)$$

for some nonnegative integer coefficients $a_{\lambda,w}$ with $\lambda \vdash \ell(w)$ in a given interval in dominance order.

Proof via an insertion algorithm like the RSK:
$\mathbf{a} = a_1 a_2 \ldots a_p \longleftrightarrow (P(\mathbf{a}), Q(\mathbf{a}))$.
$P(\mathbf{a})$ is strictly increasing in rows and columns whose reading word is a reduced word for w.
$Q(\mathbf{a})$ can be any standard tableau of the same shape as $P(\mathbf{a})$.

Corollary. Every reduced word for w_0 inserts to the same P tableau of staircase shape δ, so $|R(w_0)| = \#SYT(\delta)$.

Random Reduced Word

The formula $|R(w)| = \sum a_{\lambda,w} \# SYT(\lambda)$ gives rise to an easy way to choose a random reduced word for w using the Hook Walk Algorithm (Greene-Nijenhuis-Wilf) for random STY of shape λ.

Algorithm. Input: $w \in S_n$, Output: $a_1 a_2 \ldots a_p \in R(w)$ chosen uniformly at random.

1. Choose a P-tableau for w in proportion to $\# SYT(sh(P))$.
2. Set $\lambda = sh(P)$.
3. Loop for k from n down to 1. Choose one of the k empty cells c in λ with equal probability. Apply hook walk from c.
4. Hook walk: If c is in an outer corner of λ, place k in that cell. Otherwise, choose a new cell in the hook of c uniformly. Repeat step until c is an outer corner.

Random Reduced Word

Def. For $\mathbf{a} = a_1 a_2 \ldots a_p \in R(w)$, let $B(\mathbf{a})$ be the random variable counting the number of *braids* in $\mathbf{a}$, i.e. consecutive letters $i, i+1, i$ or $i+1, i, i+1$.

Examples. $B(321323) = 1$ and $B(232123) = 2$

Question. What is the expected value of B on $R(w)$?

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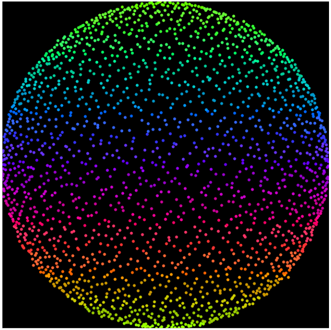
Question. What is the expected value of B on $R(w)$?

Thm.(Reiner, 2005) For all $n \geq 1$, the expected value of B on $R(w_0)$ is exactly 1.

Random Reduced Word

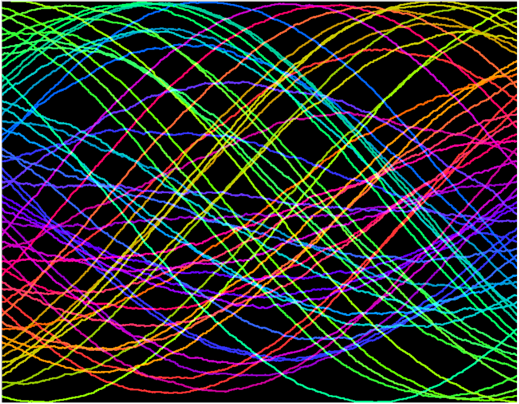
Angel-Holroyd-Romik-Virag: "Random Sorting Networks" (2007)

Conjecture. Assume $a_1 a_2 \dots a_p \in R(w_0)$ is chosen uniformly at random. The distribution of 1's in the permutation matrix for $w = s_{a_1} s_{a_2} \cdots s_{a_{p/2}}$ converges as n approaches infinity to the projected surface measure of the 2-sphere.



Random Reduced Word

Alexander Holroyd's picture of a uniformly random 2000-element sorting network (selected trajectories shown):



Macdonald's Formula

Thm. (Macdonald, 1991) For $w_0 \in S_n$,

$$\sum_{\mathbf{a} \in R(w_0)} a_1 \cdot a_2 \cdots a_{\binom{n}{2}} =$$

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Question.(Holroyd) Is there an efficient algorithm to choose a reduced word randomly with $P(a_1 a_2 \dots a_{\binom{n}{2}})$ proportional to $a_1 \cdot a_2 \cdots a_{\binom{n}{2}}$?

Consequences of Macdonald's Formula

Thm.(Young, 2014) There exists a Markov growth process using Little's bumping algorithm adding one crossing in a wiring diagram at a time to obtain a random reduced word for $w_0 \in S_n$ in $\binom{n}{2}$ steps.

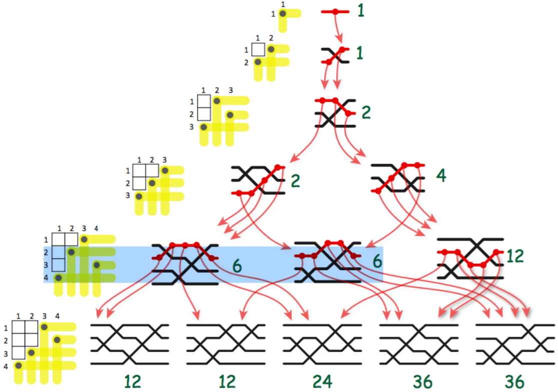
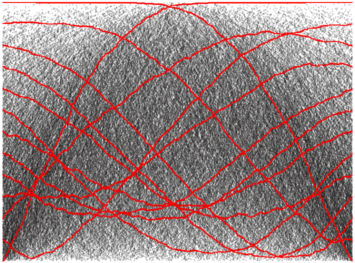


Image credit: Kristin Potter.

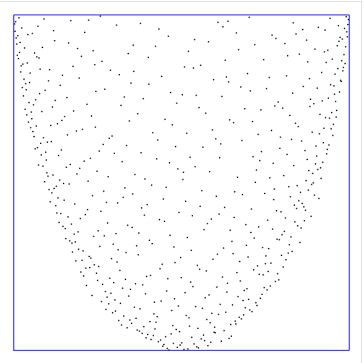
Consequences of Macdonald's Formula

The wiring diagram for a random reduced word for $w_0 \in S_{600}$ chosen with Young's growth process.



Macdonald's Formula

The permutation matrix for the product of the first half of a random reduced word for $w_0 \in S_{600}$ chosen with Young's growth process.



Macdonald's Formula

Thm.(Macdonald, 1991) For any $w \in S_n$ with $\ell(w) = p$,

$$\sum_{\mathbf{a} \in R(w)} a_1 \cdot a_2 \cdots a_p = p! \mathfrak{S}_w(1,1,1,\ldots)$$

where $\mathfrak{S}_w(1,1,1,\ldots)$ is the number of monomials in the corresponding Schubert polynomial.

Question.(Young, Fomin, Kirillov,Stanley, Macdonald, ca 1990) Is there a bijective proof of this formula?

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Question.(Young, Fomin, Kirillov,Stanley, Macdonald, ca 1990) Is there a bijective proof of this formula?

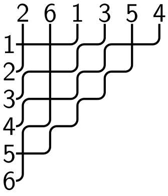
Answer. Yes! Based on joint work with Holroyd and Young, and builds on the Young's growth process.

Schubert polynomials

History. Schubert polynomials were originally defined by Lascoux-Schützenberger early 1980's. Via work of Billey-Jockusch-Stanley, Fomin-Stanley, Fomin-Kirillov, Billey-Bergeron in the early 1990's we know the following equivalent definition.

Def. For $w \in S_n$, $\mathfrak{S}_w(x_1, x_2, \dots x_n) = \sum_{D \in RP(w)} x^D$ where $RP(w)$ are the *reduced pipe dreams* for w, aka *rc-graphs*.

Example. A reduced pipe dream D for $w = [2, 6, 1, 3, 5, 4]^{-1}$ where $x^D = x_1^3 x_2 x_3 x_5$.



Bijective Proof of Macdonald's Formula

To show:

$$\sum_{\mathbf{a} \in R(w)} a_1 \cdot a_2 \cdots a_p = p! \cdot \#RP(w)$$

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Def. $b_1 b_2 \ldots b_p$ is a *bounded word* for $a_1 a_2 \ldots a_p$ provided $1 \leq b_i \leq a_i$ for each i.

Def. The pair $(\mathbf{a}, \mathbf{b}) = ((a_1 a_2 \ldots a_p), (b_1 b_2 \ldots b_p))$ is a *bounded pair* for w provided $\mathbf{a} \in R(w)$ and $\mathbf{b}$ is a bounded word for $\mathbf{a}$.

Def. A word $\mathbf{c} = c_1 c_2 \ldots c_p$ is a *sub-staircase word* provided $1 \leq c_i \leq i$ for each i.

Bijective Proof of Macdonald's Formula

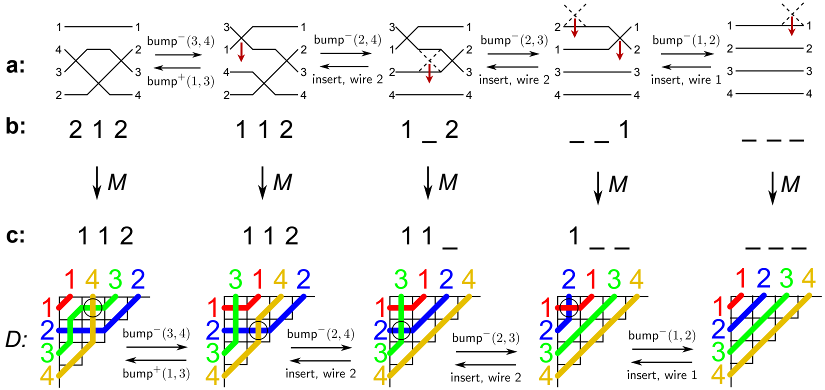
To show:

$$\sum_{\mathbf{a}\in R(w)} a_1 \cdot a_2 \cdots a_p = p! \cdot \#RP(w)$$

Want: A bijection $BP(w) \longrightarrow cD(w)$ where

- $BP(w)$:= bounded pairs for w,
- $cD(w)$:= cD-pairs for w of the form $(\mathbf{c}, D)$ where D is a reduced pipe dream for w and $\mathbf{c}$ is a sub-staircase word of the same length as w.

Bijective Proof of Macdonald's Formula



Transition Equations

Thm.(Lascoux-Schützenberger,1984) For all $w \neq id$, let $(r < s)$ be the largest pair of positions inverted in w in lexicographic order. Then,

$$\mathfrak{S}_w = x_r \mathfrak{S}_{wt_{rs}} + \sum \mathfrak{S}_{w'}$$

where the sum is over all w' such that $\ell(w) = \ell(w')$ and $w' = wt_{rs}t_{ir}$ with $0 < i < r$. Call this set $T(w)$.

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Example. If $w = [7325614]$, then $r = 5$, $s = 7$

$$\mathfrak{S}_w = x_5 \mathfrak{S}_{[7325416]} + \mathfrak{S}_{[7425316]} + \mathfrak{S}_{[7345216]}$$

So, $T(w) = \{[7425316], [7345216]\}$.

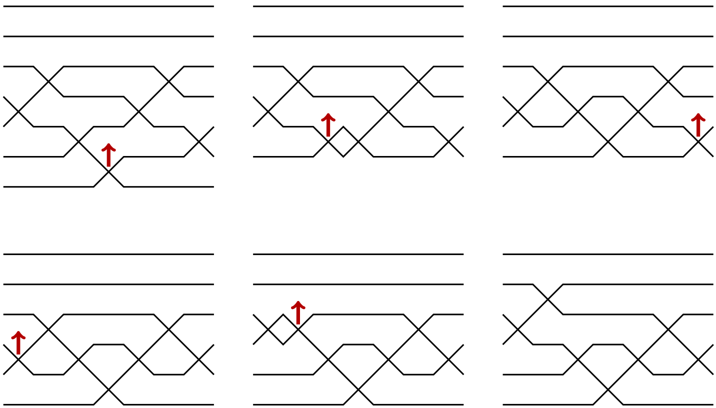
Little's Bijection

Theorem.(David Little, 2003)
There exists a bijection from $R(w)$ to $\cup_{w' \in T(w)} R(w')$ which preserves the ascent set provided $T(w)$ is nonempty.

Theorem. (Hamaker-Young, 2013) Little's bijection also preserves the Coxeter-Knuth classes and the Q-tableaux under the Edelman-Greene correspondence. Furthermore, every reduced word for any permutation with the same Q tableau is connected via Little bumps.

Little Bumps

Example. The Little bump applied to **a** $= 4356435$ in col 4.



Push and Delete operators

Let $\mathbf{a} = a_1 \dots a_k$ be a word. Define the *decrement-push*, *increment-push*, and *deletion* of $\mathbf{a}$ at column t, respectively, to be

$$\begin{aligned}
\mathcal{P}_t^- \mathbf{a} &= (a_1, \dots, a_{t-1}, a_t - 1, a_{t+1}, \dots, a_k); \\
\mathcal{P}_t^+ \mathbf{a} &= (a_1, \dots, a_{t-1}, a_t + 1, a_{t+1}, \dots, a_k); \\
\mathcal{D}_t \mathbf{a} &= (a_1, \dots, a_{t-1}, a_{t+1}, \dots, a_k);
\end{aligned}$$

Bounded Bumping Algorithm

Input: $(\mathbf{a}, \mathbf{b}, t_0, d)$, where $\mathbf{a}$ is a word that is nearly reduced at t_0, and $\mathbf{b}$ is a bounded word for $\mathbf{a}$, and $d \in \{-, +\}$.

Output: $\mathrm{Bump}^d_{t_0}(\mathbf{a}, \mathbf{b}) = (\mathbf{a}', \mathbf{b}', i, j, \text{outcome})$.

1. Initialize $\mathbf{a}' \leftarrow \mathbf{a}$, $\mathbf{b}' \leftarrow \mathbf{b}$, $t \leftarrow t_0$.
2. Push in direction d at column t, i.e. set $\mathbf{a}' \leftarrow \mathcal{P}^d_t \mathbf{a}'$ and $\mathbf{b}' \leftarrow \mathcal{P}^d_t \mathbf{b}'$.
3. If $b'_t = 0$, return $(\mathcal{D}_t \mathbf{a}', \mathcal{D}_t \mathbf{b}', \mathbf{a}'_t, t, \textit{deleted})$ and **stop**.
4. If $\mathbf{a}'$ is reduced, return $(\mathbf{a}', \mathbf{b}', \mathbf{a}'_t, t, \textit{bumped})$ and **stop**.
5. Set $t \leftarrow \mathrm{Defect}_t(\mathbf{a}')$ and **return to step 2.**

Generalizing the Transition Equation

1. We use the bounded bumping algorithm applied to the (r, s) crossing in a reduced pipe dream for w to bijectively prove

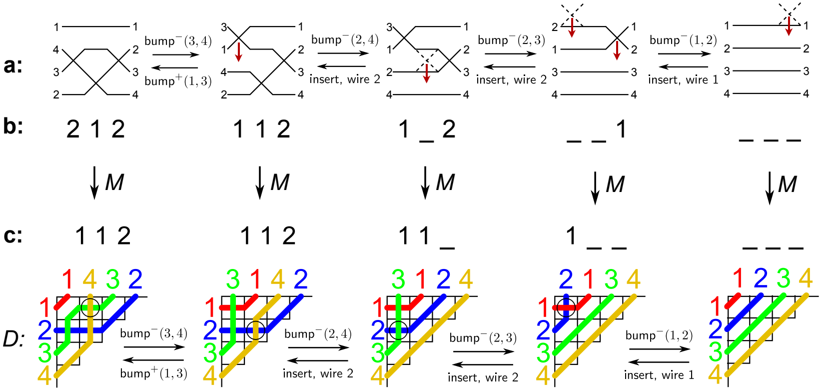
$$\mathfrak{S}_w = x_r \mathfrak{S}_{wt_{rs}} + \sum \mathfrak{S}_{w'}.$$

2. We use the bounded bumping algorithm applied to the (r, s) crossing to give a bijection

$$BP(w) \longrightarrow BP(wt_{rs}) \times [1, p] \cup \bigcup_{w' \in T(w)} BP(w').$$

Bijective Proof of Macdonald's Formula

$$\sum_{\mathbf{a} \in R(w)} a_1 \cdot a_2 \cdots a_p = p! \mathfrak{S}_w(1,1,1,\ldots)$$



q-analog of Macdonald's Formula

Def. A q-analog of any integer sequence $f_1, f_2, \ldots$ is a family of polynomials in q, $f_1(q), f_2(q), \ldots$ such that $f_i(1) = f_i$.

Examples.

- The standard q-analog of a positive integer k is $[k] = [k]_q := 1 + q + q^2 + \cdots + q^{k-1}$.
- The standard q-analog of the factorial $k!$ is defined to be $[k]_q! := [k][k-1]\cdots[1]$.

Macdonald conjectured a q-analog of his formula using $[k]$, $[k]_q!$.

q-analog of Macdonald's Formula

Theorem.(Fomin and Stanley, 1994)
Given a permutation $w \in S_n$ with $\ell(w) = p$,

$$\sum_{\mathbf{a} \in R(w)} [a_1] \cdot [a_2] \cdots [a_p] \; q^{\mathrm{comaj}(\mathbf{a})} = [p]_q! \, \mathfrak{S}_w(1, q, q^2, \ldots)$$

where

$$\mathrm{comaj}(\mathbf{a}) = \sum_{a_i < a_{i+1}} i.$$

Remarks. Our bijection respects the q-weight on each side so we get a bijective proof for this identity too. The key lemma is a generalization of Carlitz's proof that $\ell(w)$ and $\mathrm{comaj}(w)$ are equidistributed on S_n and another generalization of the Transition Equation.

Another generalization of Macdonald's formula

Fomin-Kirillov, 1997. We have the following identity of polynomials in x for the permutation $w_0 \in S_n$:

$$\sum_{\mathbf{a} \in R(w_0)} (x + a_1) \cdots (x + a_{\binom{n}{2}}) = \binom{n}{2}! \prod_{1 \le i < j \le n} \frac{2x + i + j - 1}{i + j - 1}.$$

Remarks. Our bijective proof of Macdonald's formula plus a bijection due to Serrano-Stump give a new proof of this identity answering a question posed by Fomin-Kirillov.

The right hand side is based on Proctor's formula for reverse plane partitions and Wach's characterization of Schubert polynomials for vexillary permutations.

Open Problems

Open. Is there a common generalization for the Transition Equation for Schubert polynomials, bounded pairs, and its q-analog?

Open. Is there a nice formula for $|rpp^{\lambda}(x)|$ or $[rpp^{\lambda}(x)]_q$ for an arbitrary partition λ as in the case of staircase shapes as noted in the Fomin-Kirillov Theorem?

Open. What is the analog of Macdonald's formula for Grothendieck polynomials and what is the corresponding bijection?