

# Patterns in Standard Young Tableaux

Sara Billey

University of Washington

Slides: [math.washington.edu/~billey/talks](http://math.washington.edu/~billey/talks)

Based on joint work with:

Matjaž Konvalinka and Joshua Swanson

Permutation Patterns Conference

July 10, 2018

# Outline

Background on Standard Young Tableaux

$q$ -enumeration of SYT's via major index

Distribution Question: From Combinatorics to Probability

Existence Question: New Posets on Tableaux

Unimodality Question: ???

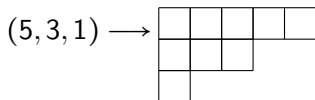
# Partitions

**Def.** A *partition* of a number  $n$  is a weakly decreasing sequence of positive integers

$$\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k > 0)$$

such that  $n = \lambda_1 + \lambda_2 + \dots + \lambda_k = |\lambda|$ . Write  $\lambda \vdash n$ .

Partitions can be visualized by their *Ferrers diagram*



The *cells* are indexed by matrix coordinates  $(i, j)$  so  $(1, 5)$  is the rightmost cell in the top row.

# Conjugate Partition

**Def.** The *conjugate* of a partition  $\lambda \vdash n$  is the partition  $\lambda' \vdash n$  whose parts count the number of cells in each column of  $\lambda$ .

$$\lambda = (5, 3, 1) = \begin{array}{|c|c|c|c|c|} \hline \square & \square & \square & \square & \square \\ \hline \square & \square & \square & & \\ \hline \square & & & & \\ \hline \end{array} \text{ and } \lambda' = (3, 2, 2, 1, 1) = \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \\ \hline \square & \square & \\ \hline \square & & \\ \hline \square & & \\ \hline \end{array}$$

The *cells* are indexed by matrix coordinates  $(i, j)$  so  $(1, 5)$  is the rightmost cell in the top row.

# Filling Partitions

**Defn.** A map from the cells of  $\lambda$  to the positive integers is a *filling* of  $\lambda$ .

1	3	7	2	8
6	1	2		
5				

**Defn.** A filling of  $\lambda \vdash n$  is *bijective* if every number in  $[n] = \{1, 2, \dots, n\}$  appears exactly once.

1	3	7	4	9
6	2	8		
5				

# Filling Partitions

**Defn.** A map from the cells of  $\lambda$  to the positive integers is a *filling* of  $\lambda$ .

1	3	7	2	8
6	1	2		
5				

**Defn.** A filling of  $\lambda \vdash n$  is *bijective* if every number in  $[n] = \{1, 2, \dots, n\}$  appears exactly once.

1	3	7	4	9
6	2	8		
5				

**Question.** How many bijective fillings are there of shape  $(5, 3, 1)$ ?

# Filling Partitions

**Defn.** A map from the cells of  $\lambda$  to the positive integers is a *filling* of  $\lambda$ .

1	3	7	2	8
6	1	2		
5				

**Defn.** A filling of  $\lambda \vdash n$  is *bijective* if every number in  $[n] = \{1, 2, \dots, n\}$  appears exactly once.

1	3	7	4	9
6	2	8		
5				

**Question.** How many bijective fillings are there of shape  $(5, 3, 1)$ ?

**Answer.**  $9! = 362,880$ . Bijection with permutations of 9.

# Standard Young Tableaux

**Defn.** A *standard Young tableaux* of shape  $\lambda$  is a bijective filling of  $\lambda$  such that every row is increasing from left to right and every column is increasing from top to bottom.

1	3	6	7	9
2	5	8		
4				

**Important Fact.** The standard Young tableaux of shape  $\lambda$ , denoted  $\text{SYT}(\lambda)$ , index a basis of the irreducible  $S_n$  representation indexed by  $\lambda$ .



# Standard Young Tableaux

**Defn.** A *standard Young tableaux* of shape  $\lambda$  is a bijective filling of  $\lambda$  such that every row is increasing from left to right and every column is increasing from top to bottom.

1	3	6	7	9
2	5	8		
4				

**Important Fact.** The standard Young tableaux of shape  $\lambda$ , denoted  $\text{SYT}(\lambda)$ , index a basis of the irreducible  $S_n$  representation indexed by  $\lambda$ .

**Question.** How many standard Young tableaux are there of shape  $(5, 3, 1)$ ?

# Standard Young Tableaux

**Defn.** A *standard Young tableaux* of shape  $\lambda$  is a bijective filling of  $\lambda$  such that every row is increasing from left to right and every column is increasing from top to bottom.

1	3	6	7	9
2	5	8		
4				

**Important Fact.** The standard Young tableaux of shape  $\lambda$ , denoted  $\text{SYT}(\lambda)$ , index a basis of the irreducible  $S_n$  representation indexed by  $\lambda$ .

**Question.** How many standard Young tableaux are there of shape  $(5, 3, 1)$ ? **Answer.**  $\#\text{SYT}(5, 3, 1) = 162$

# Standard Young Tableaux

Pause: Find all standard Young tableaux on  $(2, 2)$ .

# Counting Standard Young Tableaux

**Hook Length Formula.** (Frame-Robinson-Thrall, 1954)

If  $\lambda$  is a partition of  $n$ , then

$$\#SYT(\lambda) = \frac{n!}{\prod_{c \in \lambda} h_c}$$

where  $h_c$  is the *hook length* of the cell  $c$ , i.e. the number of cells directly to the right of  $c$  or below  $c$ , including  $c$ .

**Example.** Filling cells of  $\lambda = (5, 3, 1) \vdash 9$  by hook lengths:

7	5	4	2	1
4	2	1		
1				

So,  $\#SYT(5, 3, 1) = \frac{9!}{7 \cdot 5 \cdot 4 \cdot 2 \cdot 4 \cdot 2} = 162$ .

# Counting Standard Young Tableaux

**Hook Length Formula.** (Frame-Robinson-Thrall, 1954)

If  $\lambda$  is a partition of  $n$ , then


$$\#SYT(\lambda) = \frac{n!}{\prod_{c \in \lambda} h_c}$$

where  $h_c$  is the *hook length* of the cell  $c$ , i.e. the number of cells directly to the right of  $c$  or below  $c$ , including  $c$ .

**Example.** Filling cells of  $\lambda = (5, 3, 1) \vdash 9$  by hook lengths:

7	5	4	2	1
4	2	1		
1				

So,  $\#SYT(5, 3, 1) = \frac{9!}{7 \cdot 5 \cdot 4 \cdot 2 \cdot 4 \cdot 2} = 162$ .

**Remark.** Notable other proofs by Greene-Nijenhuis-Wilf '79 (probabilistic), Eriksson '93 (bijective), Krattenthaler '95 (bijective), Novelli -Pak -Stoyanovskii'97 (bijective), Bandlow'08, 

# $q$ -Counting Standard Young Tableaux

**Def.** The *descent set* of a standard Young tableaux  $T$ , denoted  $D(T)$ , is the set of positive integers  $i$  such that  $i + 1$  lies in a row strictly below the cell containing  $i$  in  $T$ .

The *major index* of  $T$  is the sum of its descents:

$$\text{maj}(T) = \sum_{i \in D(T)} i.$$

**Example.** The descent set of  $T$  is  $D(T) = \{1, 3, 4, 7\}$  so  $\text{maj}(T) = 15$  for  $T =$

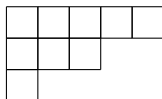
1	3	6	7	9
2	4	8		
5				

**Def.** The *major index generating function* for  $\lambda$  is

$$\text{SYT}(\lambda)^{\text{maj}}(q) := \sum_{T \in \text{SYT}(\lambda)} q^{\text{maj}(T)}$$

# $q$ -Counting Standard Young Tableaux

**Example.**  $\lambda = (5, 3, 1)$



$$\text{SYT}(\lambda)^{\text{maj}}(q) := \sum_{T \in \text{SYT}(\lambda)} q^{\text{maj}(T)} =$$

$$\begin{aligned} & q^{23} + 2q^{22} + 4q^{21} + 5q^{20} + 8q^{19} + 10q^{18} + 13q^{17} + 14q^{16} + 16q^{15} \\ & + 16q^{14} + 16q^{13} + 14q^{12} + 13q^{11} + 10q^{10} + 8q^9 + 5q^8 + 4q^7 + 2q^6 + q^5 \end{aligned}$$

Note, at  $q = 1$ , we get back 162.

# $q$ -Counting Standard Young Tableaux

**Thm.** (Lusztig-Stanley 1979) Given a partition  $\lambda \vdash n$ , say

$$\text{SYT}(\lambda)^{\text{maj}}(q) := \sum_{T \in \text{SYT}(\lambda)} q^{\text{maj}(T)} = \sum_{k \geq 0} b_{\lambda,k} q^k.$$

Then  $b_{\lambda,k} := \#\{T \in \text{SYT}(\lambda) : \text{maj}(T) = k\}$  is the number of times the irreducible  $S_n$  module indexed by  $\lambda$  appears in the decomposition of the coinvariant algebra  $\mathbb{Z}[x_1, x_2, \dots, x_n]/I_+$  in the homogeneous component of degree  $k$ .

## Comments.

- ▶ The “*fake degree sequence*” is  $(b_{\lambda,0}, b_{\lambda,1}, b_{\lambda,2}, \dots)$ .



# $q$ -Counting Standard Young Tableaux

**Thm.** (Lusztig-Stanley 1979) Given a partition  $\lambda \vdash n$ , say

$$\text{SYT}(\lambda)^{\text{maj}}(q) := \sum_{T \in \text{SYT}(\lambda)} q^{\text{maj}(T)} = \sum_{k \geq 0} b_{\lambda,k} q^k.$$

Then  $b_{\lambda,k} := \#\{T \in \text{SYT}(\lambda) : \text{maj}(T) = k\}$  is the number of times the irreducible  $S_n$  module indexed by  $\lambda$  appears in the decomposition of the coinvariant algebra  $\mathbb{Z}[x_1, x_2, \dots, x_n]/I_+$  in the homogeneous component of degree  $k$ .

## Comments.

- ▶ The “*fake degree sequence*” is  $(b_{\lambda,0}, b_{\lambda,1}, b_{\lambda,2}, \dots)$ .
- ▶ The fake degrees also appear in branching rules between symmetric groups and cyclic subgroups (Stembridge, 1989), and the degree polynomials of certain irreducible  $\text{GL}_n(\mathbb{F}_q)$ -representations (Steinberg 1951, Green 1955).

## $q$ -Counting Standard Young Tableaux

**Def.** The *descent set* of a standard Young tableaux  $T$ , denoted  $D(T)$ , is the set of positive integers  $i$  such that  $i + 1$  lies in a row strictly below the cell containing  $i$  in  $T$ .

The *major index* of  $T$  is the sum of its descents:

$$\text{maj}(T) = \sum_{i \in D(T)} i.$$

**Example.** There are 2 standard Young tableaux of shape  $(2, 2)$ :

$$S = \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array} \quad T = \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array}$$

$D(S) = \{2\}$  and  $D(T) = \{1, 3\}$  so  $\text{SYT}(\lambda)^{\text{maj}}(q) = q^2 + q^4$ .  
Represent  $q^2 + q^4$  by the vector of coefficients  $(00101)$ .

## $q$ -Counting Standard Young Tableaux

**Examples.**  $(2,2) \vdash 4$ :  $(0\ 0\ 1\ 0\ 1)$

$(5,3,1)$ :  $(00000\ 1\ 2\ 4\ 5\ 8\ 10\ 13\ 14\ 16\ 16\ 16\ 14\ 13\ 10\ 8\ 5\ 4\ 2\ 1)$

# $q$ -Counting Standard Young Tableaux

**Examples.**  $(2,2) \vdash 4$ : (0 0 1 0 1)

$(5,3,1)$ : (00000 1 2 4 5 8 10 13 14 16 16 16 14 13 10 8 5 4 2 1)

$(6,4) \vdash 10$ : (0 0 0 0 1 1 2 2 4 4 6 6 8 7 8 7 8 6 6 4 4 2 2 1 1)

$(6,6) \vdash 12$ : (0 0 0 0 0 0 1 0 1 1 2 2 4 3 5 5 7 6 9 7 9 8 9 7 9 6 7 5  
5 3 4 2 2 1 1 0 1)

$(11,5,3,1) \vdash 20$ : (1 3 8 16 32 57 99 160 254 386 576 832 1184  
1645 2255 3031 4027 5265 6811 8689 10979 13706 16959 20758  
25200 30296 36143 42734 50163 58399 67523 77470 88305 99925  
112370 125492 139307 153624 168431 183493 198778 214017  
229161 243913 258222 271780 284542 296200 306733 315853  
323571 329629 334085 336727 337662 336727 334085 329629  
323571 315853 306733 296200 284542 271780 258222 243913  
229161 214017 198778 183493 168431 153624 139307 125492  
112370 99925 88305 77470 67523 58399 50163 42734 36143  
30296 25200 20758 16959 13706 10979 8689 6811 5265 4027

# Key Questions for $\text{SYT}(\lambda)^{\text{maj}}(q)$

Recall  $\text{SYT}(\lambda)^{\text{maj}}(q) = \sum b_{\lambda,k} q^k$ .

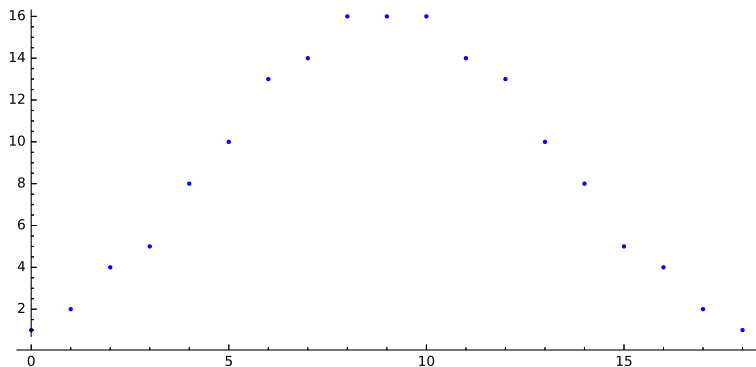
**Distribution Question.** What patterns do the coefficients in the list  $(b_{\lambda,0}, b_{\lambda,1}, \dots)$  exhibit?

**Existence Question.** For which  $\lambda, k$  does  $b_{\lambda,k} = 0$  ?

**Unimodality Question.** For which  $\lambda$ , are the coefficients of  $\text{SYT}(\lambda)^{\text{maj}}(q)$  *unimodal*, meaning

$$b_{\lambda,0} \leq b_{\lambda,1} \leq \dots \leq b_{\lambda,m} \geq b_{\lambda,m+1} \geq \dots?$$

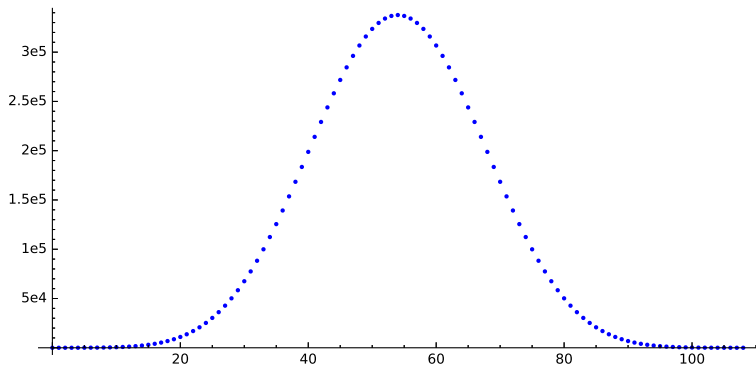
# Visualizing Major Index Generating Functions



Visualizing the coefficients of  $\text{SYT}(5, 3, 1)^{\text{maj}}(q)$ :

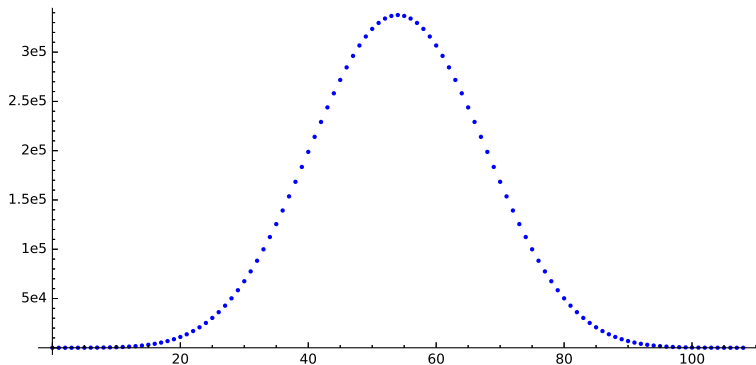
(1, 2, 4, 5, 8, 10, 13, 14, 16, 16, 16, 14, 13, 10, 8, 5, 4, 2, 1)

# Visualizing Major Index Generating Functions



Visualizing the coefficients of  $\text{SYT}(11, 5, 3, 1)^{\text{maj}}(q)$ .

# Visualizing Major Index Generating Functions

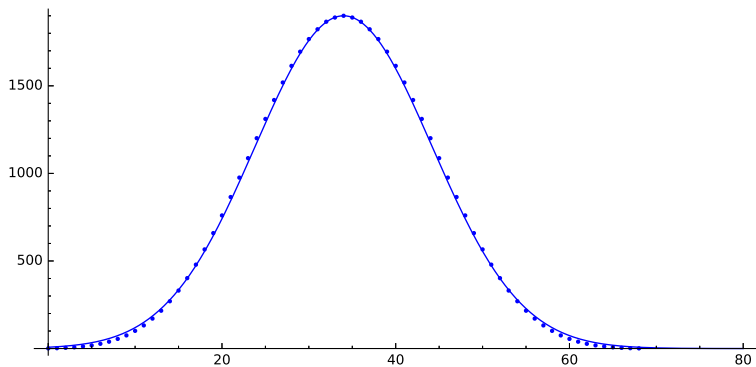


Visualizing the coefficients of  $\text{SYT}(11, 5, 3, 1)^{\text{maj}}(q)$ .

**Question.** What type of curve is that?

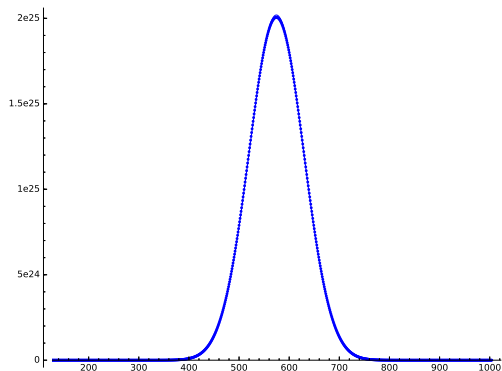


# Visualizing Major Index Generating Functions



Visualizing the coefficients of  $\text{SYT}(10, 6, 1)^{\text{maj}}(q)$  along with the Normal distribution with  $\mu = 34$  and  $\sigma^2 = 98$ .

# Visualizing Major Index Generating Functions



Visualizing the coefficients of  $\text{SYT}(8, 8, 7, 6, 5, 5, 5, 2, 2)^{\text{maj}}(q)$

# “Fast” Computation of $\text{SYT}(\lambda)^{\text{maj}}(q)$

**Thm.** (Stanley’s  $q$ -analog of the Hook Length Formula for  $\lambda \vdash n$ )

$$\text{SYT}(\lambda)^{\text{maj}}(q) = \frac{q^{b(\lambda)} [n]_q!}{\prod_{c \in \lambda} [h_c]_q}$$

where

- ▶  $b(\lambda) := \sum (i-1)\lambda_i$
- ▶  $h_c$  is the hook length of the cell  $c$
- ▶  $[n]_q := 1 + q + \dots + q^{n-1} = \frac{q^n - 1}{q - 1}$
- ▶  $[n]_q! := [n]_q [n-1]_q \dots [1]_q$

## “Fast” Computation of $\text{SYT}(\lambda)^{\text{maj}}(q)$

**Thm.** (Stanley’s  $q$ -analog of the Hook Length Formula for  $\lambda \vdash n$ )

$$\text{SYT}(\lambda)^{\text{maj}}(q) = \frac{q^{b(\lambda)} [n]_q!}{\prod_{c \in \lambda} [h_c]_q}$$

where

- ▶  $b(\lambda) := \sum (i-1)\lambda_i$
- ▶  $h_c$  is the hook length of the cell  $c$
- ▶  $[n]_q := 1 + q + \dots + q^{n-1} = \frac{q^n - 1}{q - 1}$
- ▶  $[n]_q! := [n]_q [n-1]_q \dots [1]_q$

**The Trick.** Each  $q$ -integer  $[n]_q$  factors into a product of *cyclotomic polynomials*  $\Phi_d(q)$ ,

$$[n]_q = 1 + q + \dots + q^{n-1} = \prod_{d|n} \Phi_d(q).$$

Cancel all of the factors from the denominator of  $\text{SYT}(\lambda)^{\text{maj}}(q)$  from the numerator, and then expand the remaining product.

# Corollaries of Stanley's formula

**Thm.** (Stanley's  $q$ -analog of the Hook Length Formula for  $\lambda \vdash n$ )

$$\text{SYT}(\lambda)^{\text{maj}}(q) = \frac{q^{b(\lambda)} [n]_q!}{\prod_{c \in \lambda} [h_c]_q}$$

## Corollaries.

1.  $\text{SYT}(\lambda)^{\text{maj}}(q) = \text{SYT}(\lambda')^{\text{maj}}(q)$ .
2. The coefficients of  $\text{SYT}(\lambda)^{\text{maj}}(q)$  are symmetric.
3. There is a unique min-maj and max-maj tableau of shape  $\lambda$ .

# Min-Maj and Max-Maj Tableaux

**Example.** The *min-maj* and *max-maj* tableaux for  $(6, 4, 3, 3, 1)$ .

1	3	4	11	16	17
2	6	7	15		
5	9	10			
8	13	14			
12					

$$b(\lambda) = \sum (i-1)\lambda_i = 23$$

1	2	3	5	9	13
4	6	10	14		
7	11	15			
8	12	16			
17					

$$\binom{17}{2} - b(\lambda') = 109$$

# Converting $q$ -Enumeration to Discrete Probability

Vic Reiner's Quote:

*"If we can count it, we should also try to  $q$ -count it."*

I say:

*"If we can  $q$ -count it, we should try to probabalize it."*

# Converting $q$ -Enumeration to Discrete Probability

Vic Reiner's Quote:

*"If we can count it, we should also try to  $q$ -count it."*

I say:

*"If we can  $q$ -count it, we should try to probabalize it."*

If  $f(q) = a_0 + a_1q + a_2q^2 + \cdots + a_nq^n$  where  $a_i$  are nonnegative integers, then construct the random variable  $X_f$  with discrete probability distribution

$$\mathbb{P}(X_f = k) = \frac{a_k}{\sum_j a_j} = \frac{a_k}{f(1)}.$$

Now, if  $f$  is part of a family of  $q$ -analogs, we can study the limiting distributions.



# Converting $q$ -Enumeration to Discrete Probability

**Example.** For  $\text{SYT}(\lambda)^{\text{maj}}(q) = \sum b_{\lambda,k} q^k$ , define the integer random variable  $X_\lambda[\text{maj}]$  with discrete probability distribution

$$\mathbb{P}(X_\lambda[\text{maj}] = k) = \frac{b_{\lambda,k}}{|\text{SYT}(\lambda)|}.$$

We claim the distribution of  $X_\lambda$  “usually” is approximately normal for most shapes  $\lambda$ . Let’s make that precise!

# Standardization

**Thm.**(Adin-Roichman, 2001)

For any partition  $\lambda$ , the mean and variance of  $X_\lambda[\text{maj}]$  are

$$\mu_\lambda = \frac{\binom{|\lambda|}{2} - b(\lambda') + b(\lambda)}{2} = b(\lambda) + \frac{1}{2} \left[ \sum_{j=1}^{|\lambda|} j - \sum_{c \in \lambda} h_c \right],$$

and

$$\sigma_\lambda^2 = \frac{1}{12} \left[ \sum_{j=1}^{|\lambda|} j^2 - \sum_{c \in \lambda} h_c^2 \right].$$

**Def.** The *standardization* of  $X_\lambda[\text{maj}]$  is

$$X_\lambda^*[\text{maj}] = \frac{X_\lambda[\text{maj}] - \mu_\lambda}{\sigma_\lambda}.$$

So  $X_\lambda^*[\text{maj}]$  has mean 0 and variance 1 for any  $\lambda$ .

# Asymptotic Normality

**Def.** Let  $X_1, X_2, \dots$  be a sequence of real-valued random variables with standardized cumulative distribution functions  $F_1(t), F_2(t), \dots$ . The sequence is *asymptotically normal* if

$$\forall t \in \mathbb{R}, \quad \lim_{n \rightarrow \infty} F_n(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^t e^{-x^2/2} = \mathbb{P}(N < t)$$

where  $N$  is a Normal random variable with mean 0 and variance 1.

# Asymptotic Normality

**Def.** Let  $X_1, X_2, \dots$  be a sequence of real-valued random variables with standardized cumulative distribution functions  $F_1(t), F_2(t), \dots$ . The sequence is *asymptotically normal* if

$$\forall t \in \mathbb{R}, \quad \lim_{n \rightarrow \infty} F_n(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^t e^{-x^2/2} = \mathbb{P}(N < t)$$

where  $N$  is a Normal random variable with mean 0 and variance 1.

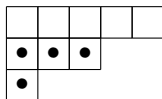
**Question.** In what way can a sequence of partitions approach infinity?

# The Aft Statistic

**Def.** Given a partition  $\lambda = (\lambda_1, \dots, \lambda_k) \vdash n$ , let

$$\text{aft}(\lambda) := n - \max\{\lambda_1, k\}.$$

**Example.**  $\lambda = (5, 3, 1)$  then  $\text{aft}(\lambda) = 4$ .



Look it up: [Aft is now on FindStat as St001214](#)

# Distribution Question: From Combinatorics to Probability

**Thm.** (Billey-Konvalinka-Swanson, 2018+)

Suppose  $\lambda^{(1)}, \lambda^{(2)}, \dots$  is a sequence of partitions, and let  $X_N := X_{\lambda^{(N)}}[\text{maj}]$  be the corresponding random variables for the maj statistic. Then, the sequence  $X_1, X_2, \dots$  is asymptotically normal if and only if  $\text{aft}(\lambda^{(N)}) \rightarrow \infty$  as  $N \rightarrow \infty$ .

# Distribution Question: From Combinatorics to Probability

**Thm.** (Billey-Konvalinka-Swanson, 2018+)

Suppose  $\lambda^{(1)}, \lambda^{(2)}, \dots$  is a sequence of partitions, and let  $X_N := X_{\lambda^{(N)}}[\text{maj}]$  be the corresponding random variables for the maj statistic. Then, the sequence  $X_1, X_2, \dots$  is asymptotically normal if and only if  $\text{aft}(\lambda^{(N)}) \rightarrow \infty$  as  $N \rightarrow \infty$ .

**Question.** What happens if  $\text{aft}(\lambda^{(N)})$  does not go to infinity as  $N \rightarrow \infty$ ?

# Distribution Question: From Combinatorics to Probability

**Thm.** (Billey-Konvalinka-Swanson, 2018+)

Let  $\lambda^{(1)}, \lambda^{(2)}, \dots$  be a sequence of partitions. Then  $(X_{\lambda^{(N)}}[\text{maj}]^*)$  converges in distribution if and only if

- (i)  $\text{aft}(\lambda^{(N)}) \rightarrow \infty$ ; or
- (ii)  $|\lambda^{(N)}| \rightarrow \infty$  and  $\text{aft}(\lambda^{(N)})$  is eventually constant; or
- (iii) the distribution of  $X_{\lambda^{(N)}}^*[\text{maj}]$  is eventually constant.

The limit law is  $\mathcal{N}(0, 1)$  in case (i),  $\Sigma_M^*$  in case (ii), and discrete in case (iii).

Here  $\Sigma_M$  denotes the sum of  $M$  independent identically distributed uniform  $[0, 1]$  random variables, known as the Irwin–Hall distribution or the *uniform sum distribution*.



# Proof ideas: Characterize the Moments and Cumulants

## Definitions.

- ▶ For  $d \in \mathbb{Z}_{\geq 0}$ , the *dth moment*

$$\mu_d := \mathbb{E}[X^d]$$

- ▶ The *moment-generating function* of  $X$  is

$$M_X(t) := \mathbb{E}[e^{tX}] = \sum_{d=0}^{\infty} \mu_d \frac{t^d}{d!},$$

- ▶ The *cumulants*  $\kappa_1, \kappa_2, \dots$  of  $X$  are defined to be the coefficients of the exponential generating function

$$K_X(t) := \sum_{d=1}^{\infty} \kappa_d \frac{t^d}{d!} := \log M_X(t) = \log \mathbb{E}[e^{tX}].$$

# Nice Properties of Cumulants

1. (*Familiar Values*) The first two cumulants are  $\kappa_1 = \mu$ , and  $\kappa_2 = \sigma^2$ .
2. (*Shift Invariance*) The second and higher cumulants of  $X$  agree with those for  $X - c$  for any  $c \in \mathbb{R}$ .
3. (*Homogeneity*) The  $d$ th cumulant of  $cX$  is  $c^d \kappa_d$  for  $c \in \mathbb{R}$ .
4. (*Additivity*) The cumulants of the sum of *independent* random variables are the sums of the cumulants.
5. (*Polynomial Equivalence*) The cumulants and moments are determined by polynomials in the other sequence.

## Examples of Cumulants and Moments

**Example.** Let  $X = \mathcal{N}(\mu, \sigma^2)$  be the normal random variable with mean  $\mu$  and variance  $\sigma^2$ . Then the cumulants are

$$\kappa_d = \begin{cases} \mu & d = 1, \\ \sigma^2 & d = 2, \\ 0 & d \geq 3. \end{cases}$$

and for  $d > 1$ ,

$$\mu_d = \begin{cases} 0 & \text{if } d \text{ is odd,} \\ \sigma^d (d-1)!! & \text{if } d \text{ is even.} \end{cases}$$

**Example.** For a Poisson random variable  $X$  with mean  $\mu$ , the cumulants are all  $\kappa_d = \mu$ , while the moments are  $\mu_d = \sum_{i=1}^d \mu^i S_{i,d}$ .

# Cumulants for Major Index Generating Functions

**Thm.** (Billey-Konvalinka-Swanson, 2018+)

Let  $\lambda \vdash n$  and  $d \in \mathbb{Z}_{>1}$ . We have

$$\kappa_d^\lambda = \frac{B_d}{d} \left[ \sum_{j=1}^n j^d - \sum_{c \in \lambda} h_c^d \right] \quad (1)$$

where  $B_0, B_1, B_2, \dots = 1, \frac{1}{2}, \frac{1}{6}, 0, -\frac{1}{30}, 0, \frac{1}{42}, 0, \dots$  are the Bernoulli numbers (OEIS A164555 / OEIS A027642).

**Remark.** We use this theorem to prove that as  $n$  approaches infinity the standardized cumulants for  $d \geq 3$  all go to 0 proving the Asymptotic Normality Theorem.

# Cumulants for Major Index Generating Functions

**Thm.** (Billey-Konvalinka-Swanson, 2018+)

Let  $\lambda \vdash n$  and  $d \in \mathbb{Z}_{>1}$ . We have

$$\kappa_d^\lambda = \frac{B_d}{d} \left[ \sum_{j=1}^n j^d - \sum_{c \in \lambda} h_c^d \right] \quad (1)$$

where  $B_0, B_1, B_2, \dots = 1, \frac{1}{2}, \frac{1}{6}, 0, -\frac{1}{30}, 0, \frac{1}{42}, 0, \dots$  are the Bernoulli numbers (OEIS A164555 / OEIS A027642).

**Remark.** We use this theorem to prove that as  $n$  approaches infinity the standardized cumulants for  $d \geq 3$  all go to 0 proving the Asymptotic Normality Theorem.

**Remark.** Note,  $\kappa_2^\lambda$  is exactly the Adin-Roichman variance formula.

## $q$ -Enumeration to Probability

**Thm.** (Chen–Wang–Wang-2008 and Hwang–Zacharovas-2015)  
Suppose  $\{a_1, \dots, a_m\}$  and  $\{b_1, \dots, b_m\}$  are multisets of positive integers such that

$$f(q) = \frac{\prod_{j=1}^m [a_j]_q}{\prod_{j=1}^m [b_j]_q} = \sum c_k q^k \in \mathbb{Z}_{\geq 0}[q]$$

Let  $X$  be a discrete random variable with  $\mathbb{P}(X = k) = c_k/f(1)$ .  
Then the  $d$ th cumulant of  $X$  is

$$\kappa_d = \frac{B_d}{d} \sum_{j=1}^m (a_j^d - b_j^d)$$

where  $B_d$  is the  $d$ th Bernoulli number (with  $B_1 = \frac{1}{2}$ ).

**Example.** This theorem applies to

$$\text{SYT}(\lambda)^{\text{maj}}(q) := \sum_{T \in \text{SYT}(\lambda)} q^{\text{maj}(T)} = \frac{q^{b(\lambda)} [n]_q!}{\prod_{c \in \lambda} [h_c]_q}$$

# Corollaries of the Distribution Theorem

1. Asymptotic normality also holds for block diagonal skew shapes with  $aft$  going to infinity.
2. New proof of asymptotic normality of  $[n]_q! = \sum_{w \in \mathcal{S}_n} q^{\text{maj}(w)} = \sum_{w \in \mathcal{S}_n} q^{\text{inv}(w)}$  due to Feller (1944).
3. New proof of asymptotic normality of  $q$ -multinomial coefficients due to Diaconis (1988), Canfield-Jansen-Zeilberger (2011).
4. New proof of asymptotic normality of  $q$ -Catalan numbers due to Chen-Wang-Wang(2008).

## Corollaries of the Distribution Theorem

1. Asymptotic normality also holds for block diagonal skew shapes with  $aft$  going to infinity.
2. New proof of asymptotic normality of  $[n]_q! = \sum_{w \in S_n} q^{\text{maj}(w)} = \sum_{w \in S_n} q^{\text{inv}(w)}$  due to Feller (1944).
3. New proof of asymptotic normality of  $q$ -multinomial coefficients due to Diaconis (1988), Canfield-Jansen-Zeilberger (2011).
4. New proof of asymptotic normality of  $q$ -Catalan numbers due to Chen-Wang-Wang(2008).

**Question.** Using Pak-Panova-Morales's  $q$ -hook length formula, can we prove an asymptotic normality for all skew shapes?



## Existence Question

Recall  $\text{SYT}(\lambda)^{\text{maj}}(q) = \sum_{T \in \text{SYT}(\lambda)} q^{\text{maj}(T)} = \sum b_{\lambda,k} q^k$ .

**Existence Question.** For which  $\lambda, k$  does  $b_{\lambda,k} = 0$  ?

## Existence Question

Recall  $\text{SYT}(\lambda)^{\text{maj}}(q) = \sum_{T \in \text{SYT}(\lambda)} q^{\text{maj}(T)} = \sum b_{\lambda,k} q^k$ .

**Existence Question.** For which  $\lambda, k$  does  $b_{\lambda,k} = 0$  ?

**Cor of Stanley's formula.** For every  $\lambda \vdash n \geq 1$  there is a unique tableau with minimal major index  $b(\lambda)$  and a unique tableau with maximal major index  $\binom{n}{2} - b(\lambda')$ . These two agree for shapes consisting of one row or one column, and otherwise they are distinct.

# Patterns on Tableaux

**Example.** The min-maj and max-maj tableaux for  $(6, 4, 3, 3, 1)$ .

1	3	4	11	16	17
2	6	7	15		
5	9	10			
8	13	14			
12					

$$b(\lambda) = \sum (i-1)\lambda_i = 23$$

1	2	3	5	9	13
4	6	10	14		
7	11	15			
8	12	16			
17					

$$\binom{17}{2} - b(\lambda') = 109$$

## Existence Question

Recall  $\text{SYT}(\lambda)^{\text{maj}}(q) = \sum_{T \in \text{SYT}(\lambda)} q^{\text{maj}(T)} = \sum b_{\lambda,k} q^k$ .

**Existence Question.** For which  $\lambda, k$  does  $b_{\lambda,k} = 0$  ?

**Cor of Stanley's formula.** The coefficient of  $q^{b(\lambda)+1}$  in  $\text{SYT}(\lambda)^{\text{maj}}(q) = 0$  if and only if  $\lambda$  is a rectangle.

If  $\lambda$  is a rectangle with more than one row and column, then coefficient of  $q^{b(\lambda)+2}$  is 1.

**Question.** Are there other internal zeros?

# Classifying All Nonzero Fake Degrees

**Thm.** (Billey-Konvalinka-Swanson, 2018+)

For any partition  $\lambda$  which is not a rectangle,

$$\text{SYT}(\lambda)^{\text{maj}}(q) := \sum_{T \in \text{SYT}(\lambda)} q^{\text{maj}(T)}$$

as no internal zeros. If  $\lambda$  is a rectangle with at least two rows and columns,  $\text{SYT}(\lambda)^{\text{maj}}(q)$  has exactly one internal zero at  $b(\lambda) + 1$  up to symmetry.

**Cor.** The irreducible  $S_n$ -module indexed by  $\lambda$  appears in the decomposition of the degree  $k$  component of the coinvariant algebra if and only if  $b_{\lambda,k} > 0$  as characterized above.

**Acknowledgment.** Our motivation for this project came from a conjecture of Sheila Sundaram's which was solved by Josh Swanson on the zeros of the maj-mod- $n$  generating function on standard Young tableaux.

# Exceptional Tableaux

**Def.** Let  $\mathcal{E}(\lambda)$  denote the set of *exceptional* tableaux of shape  $\lambda$  consisting of the following elements:

- (i) For all  $\lambda$ , the max-maj tableau for  $\lambda$ .
- (ii) If  $\lambda$  is a rectangle, the min-maj tableau for  $\lambda$ .
- (iii) If  $\lambda$  is a rectangle with at least two rows and columns, the unique tableau in  $\text{SYT}(\lambda)$  with maj equal to  $\binom{n}{2} - b(\lambda') - 2$ .

**Example.**  $\mathcal{E}(555)$  has the following three elements:

1	2	3
4	5	6
7	8	9

1	2	7
3	5	8
4	6	9

1	4	7
2	5	8
3	6	9

# Major Index Increment Map

**Proof Outline.** We give an explicit map

$$\phi : \text{SYT}(\lambda) - \mathcal{E}(\lambda) \longrightarrow \text{SYT}(\lambda)$$

such that

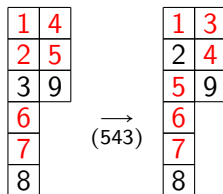
1.  $\text{maj}(\phi(T)) = \text{maj}(T) + 1$ ,
2. the descent set of  $D(T)$  and  $D(\phi(T))$  are “close”.

Internal Zeros Classification Theorem now follows by starting at the minimal maj tableau in  $\text{SYT}(\lambda) - \mathcal{E}(\lambda)$  and applying  $\phi$  recursively until it hits a tableaux in  $\mathcal{E}(\lambda)$ .

# Major Index Increment Map

**Pattern Inspired Approach.** For each  $T \in \text{SYT}(\lambda) - \mathcal{E}(\lambda)$ , identify a permutation  $\sigma$  such that  $\sigma \cdot T = T'$  is in  $\text{SYT}(\lambda)$  and  $\text{maj}(T') = \text{maj}(T) + 1$ .

**Example.**



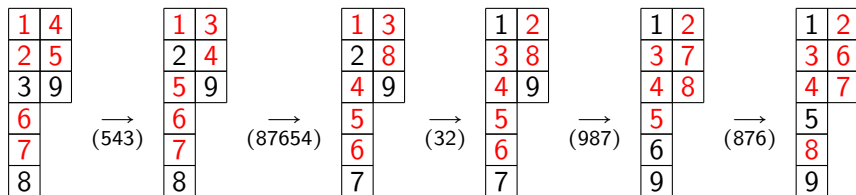
$$D(T) = \{1, 2, 4, 5, 6, 7\} \longrightarrow D(T') = \{1, 3, 4, 5, 6, 7\}$$



# Major Index Increment Map

**Pattern Inspired Approach.** For each  $T \in \text{SYT}(\lambda) \setminus \text{maxmaj}(\lambda)$ , identify a permutation  $\sigma$  such that  $\sigma \cdot T = T'$  is in  $\text{SYT}(\lambda)$  and  $\text{maj}(T') = \text{maj}(T) + 1$ .

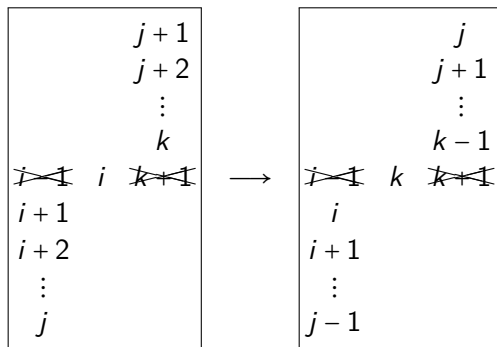
## More Examples.



$D(T): 2 \rightarrow 3, \quad 7 \rightarrow 8, \quad 1 \rightarrow 2, \quad 6 \rightarrow 7, \quad 5 \rightarrow 6$

# Patterns on Tableaux

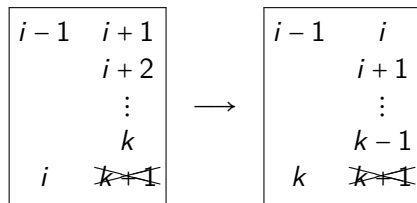
**Rotation Rule.** If there exists  $i < j < k$ , such that the *consecutive values*  $[i, k]$  follow the *descent/exclusion pattern*



then the descent set on the left contains  $j-1$  and the one on the right contains  $j$ , otherwise all other descents are the same.

# Patterns on Tableaux

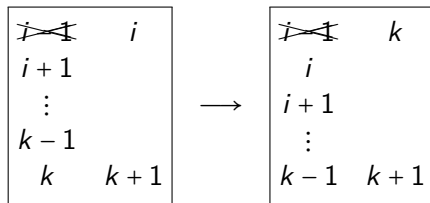
**Rotation Rule.** If there exists  $i = j < k$ , such that the values  $[i, k]$  follow the descent/exclusion pattern



then the descent set on the left contains  $j-1$  and the one on the right contains  $j$ , otherwise all other descents are the same.

# Patterns on Tableaux

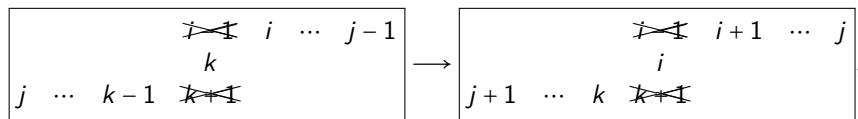
**Rotation Rule.** If there exists  $i < j = k$ , such that the values  $[i, k]$  follow the descent/exclusion pattern



then the descent set on the left contains  $j-1$  and the one on the right contains  $j$ , otherwise all other descents are the same.

# Patterns on Tableaux

**Dual Rotation Rule.** If there exists  $i < j < k$ , such that the values  $[i, k]$  follow the descent/exclusion pattern



then the descent set on the left contains  $j-1$  and the one on the right contains  $j$ , otherwise all other descents are the same.

# Patterns on Tableaux

**Fact.** Almost all standard Young tableaux admit some rotation.

**Example.** Among the 81,081 tableaux in  $\text{SYT}(5, 4, 4, 2)$ , there are only 24 (i.e., 0.03%) on which we cannot apply any rotation rule.

# Patterns on Tableaux

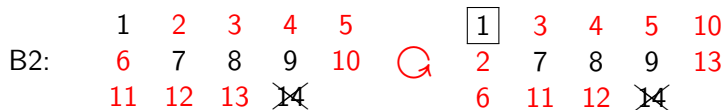
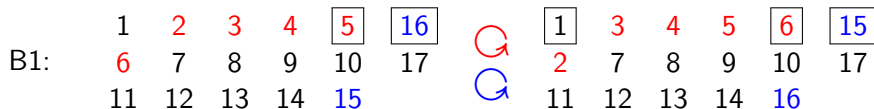
**Fact.** Almost all standard Young tableaux admit some rotation.

**Example.** Among the 81,081 tableaux in  $\text{SYT}(5, 4, 4, 2)$ , there are only 24 (i.e., 0.03%) on which we cannot apply any rotation rule.

**Question.** What about the tableaux which don't admit any rotation rules?

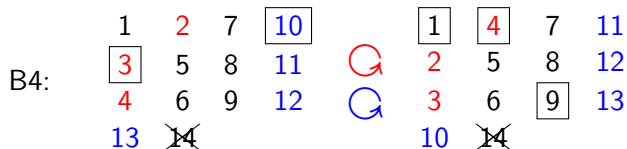
# Block Rules

**Five More Block Rules.** Adding a descent at 1, plus possibly other mutations.





# Block Rules



## Block Rules

B5:

1	2		<span style="border: 1px solid black; padding: 2px;">1</span>	6
3	6		2	7
4	7	Q	3	8
5	8		4	9
9	<del>10</del>		5	<del>10</del>

**Proof Completion by Cases.** Every tableaux which is not exceptional and avoids

$$\begin{array}{cccc} 1 & 2 & \dots & i \\ i+1 & z+1 & & \\ i+2 & & & \\ \vdots & & & \\ z & & & \end{array}$$

admits a rotation rule. All other non-exceptional tableaux admit a block rule or a rotation rule.

# Strong Poset on $\text{SYT}(\lambda)$

**Def.** The *Strong SYT Poset*  $P(\lambda)$  on either

$$\text{SYT}(\lambda) \setminus \{\text{minmaj}(\lambda), \text{maxmaj}(\lambda)\}$$

if  $\lambda$  is a rectangle with at least two rows and columns, or  $\text{SYT}(\lambda)$  otherwise, is the transitive closure of the covering relations given by all applicable rotation rules, block rules, and inverse-transpose block rules, each increasing  $\text{maj}$  by 1.

**Corollary.** As a poset,  $P(\lambda)$  is ranked according to  $\text{maj}(T)$  and has a unique minimal and maximal element.

# Weak Poset on $\text{SYT}(\lambda)$

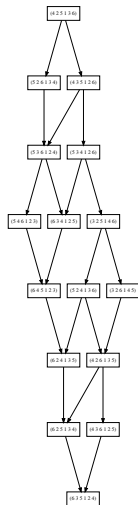
**Def.** The *Weak SYT Poset*  $Q(\lambda)$  on either

$$\text{SYT}(\lambda) \setminus \{\text{minmaj}(\lambda), \text{maxmaj}(\lambda)\}$$

if  $\lambda$  is a rectangle with at least two rows and columns, or  $\text{SYT}(\lambda)$  otherwise, is the transitive closure of the relations given by  $T < \phi(T)$  and the inverse-transpose of these rules.

**Corollary.** As a poset,  $Q(\lambda)$  is ranked according to  $\text{maj}(T)$  and has a unique minimal and maximal element.

# Strong and Weak Poset on SYT(3,2,1)



Strong



Weak

# Unimodality Question

**Conjecture.** The polynomial  $\text{SYT}^{\text{maj}}(q)$  is unimodal if  $\lambda$  has at least 4 corners. If  $\lambda$  has 3 corners or fewer, then  $\text{SYT}^{\text{maj}}(q)$  is unimodal except when  $\lambda$  or  $\lambda'$  is among the following partitions:

1. Any partition of rectangle shape that has more than one row and column.
2. Any partition of the form  $(k, 2)$  with  $k \geq 4$  and  $k$  even.
3. Any partition of the form  $(k, 4)$  with  $k \geq 6$  and  $k$  even.
4. Any partition of the form  $(k, 2, 1, 1)$  with  $k \geq 2$  and  $k$  even.
5. Any partition of the form  $(k, 2, 2)$  with  $k \geq 6$ .
6. Any partition on the list of 40 special exceptions of size at most 28.

# Unimodality Question

## Special Exceptions.

(3, 3, 2), (4, 2, 2), (4, 4, 2), (4, 4, 1, 1),  
(5, 3, 3), (7, 5), (6, 2, 1, 1, 1, 1),  
(5, 5, 2), (5, 5, 1, 1), (5, 3, 2, 2), (4, 4, 3, 1),  
(4, 4, 2, 2), (7, 3, 3), (8, 6), (6, 6, 2),  
(6, 6, 1, 1), (5, 5, 2, 2), (5, 3, 3, 3), (4, 4, 4, 2),  
(11, 5), (10, 6), (9, 7), (7, 7, 2),  
(7, 7, 1, 1), (6, 6, 4), (6, 6, 1, 1, 1, 1), (6, 5, 5),  
(5, 5, 3, 3), (12, 6), (11, 7), (10, 8),  
(15, 5), (14, 6), (11, 9), (16, 6), (12, 10), (18, 6),  
(14, 10), (20, 6), (22, 6).

# Local Limit Conjecture

**Conjecture.** Let  $\lambda \vdash n > 25$ . Uniformly for all  $n$  and for all integers  $k$ , we have

$$|\mathbb{P}(X_\lambda[\text{maj}] = k) - N(k; \mu_\lambda, \sigma_\lambda)| = O\left(\frac{1}{\sigma_\lambda \text{aft}(\lambda)}\right)$$

where  $N(k; \mu_\lambda, \sigma_\lambda)$  is the density function for the normal distribution with mean  $\mu_\lambda$  and variance  $\sigma_\lambda$ .

The conjecture has been verified for  $n \leq 50$  and  $\text{aft}(\lambda) > 1$ .

Up to  $n = 50$ , the constant  $1/9$  works.

At  $n = 50$ ,  $1/10$  does not.



Many Thanks!

P	E	R	M	U	T	A	T	I	O	N
A	2									
T	0									
T	1									
E	8									
R										
N										
S										