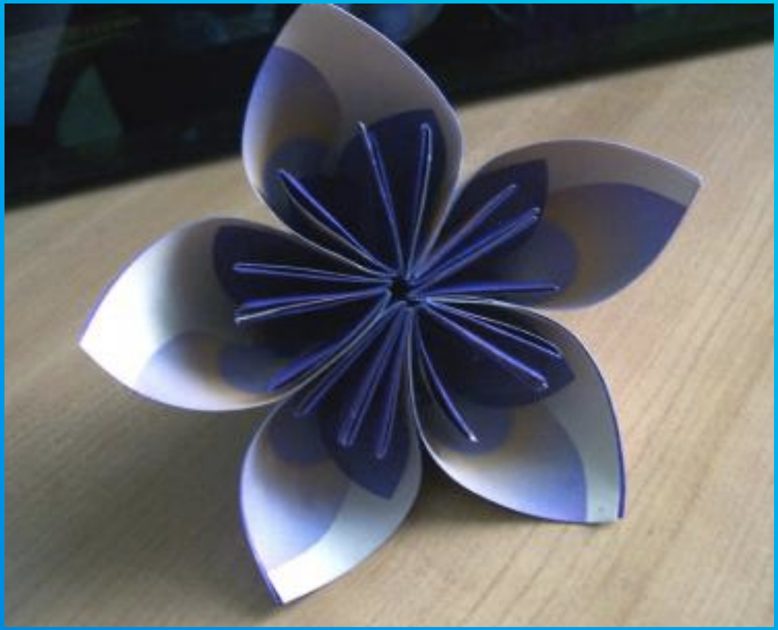
Math inspired by Origami



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Overview

Math: The art of formal thinking, patterns, and quantitative expression

Origami: The art of paper folding

Experiment: folding paper and marking points

Theorem: an existence problem and a beautiful solution

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What is Math?

Math: The art of formal thinking, patterns, and quantitative expression.

Examples: formulas, theorems, algorithms, geometrical patterns, special numerical sequences, statistics, mathematical models…

Mathematicians create new ideas every day.

Example in Math

Statement:

If a right triangle has sides a,b,c, then $a^2 + b^2 = c^2$.

Question: True or false?

Answer: True if $a,b<c$. False if $c<a$ or $c<b$.

Can you name that statement?

Example in Math

Statement: Pythagorean Theorem

If a right triangle has sides a,b,c, then $a^2 + b^2 = c^2$.

Question: True or false?

Answer: True if $a,b<c$. False if $c<a$ or $c<b$.

Example in Math

Statement:

There exist positive integers a,b,c such that $a^2+b^2=c^2$.

Question: True or false?

Answer: True $a=3,b=4,c=5$ (Pythagorean triples)

Example in Math

Statement:

There exists a rectangular box with side lengths given by 3 positive integers a,b,c such that the distance between any two corners is an integer.

Question: True or false?

Answer: Unknown, this is a current research. Its called the Perfect Cuboid Problem.

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What is Origami?

Origami: The art of folding flat objects into small 3-dimensional shapes.

Examples: folding paper, folding airbags, folding space telescopes, making small replacement heart stents.

Origami artists create new shapes every day.

Example of Origami

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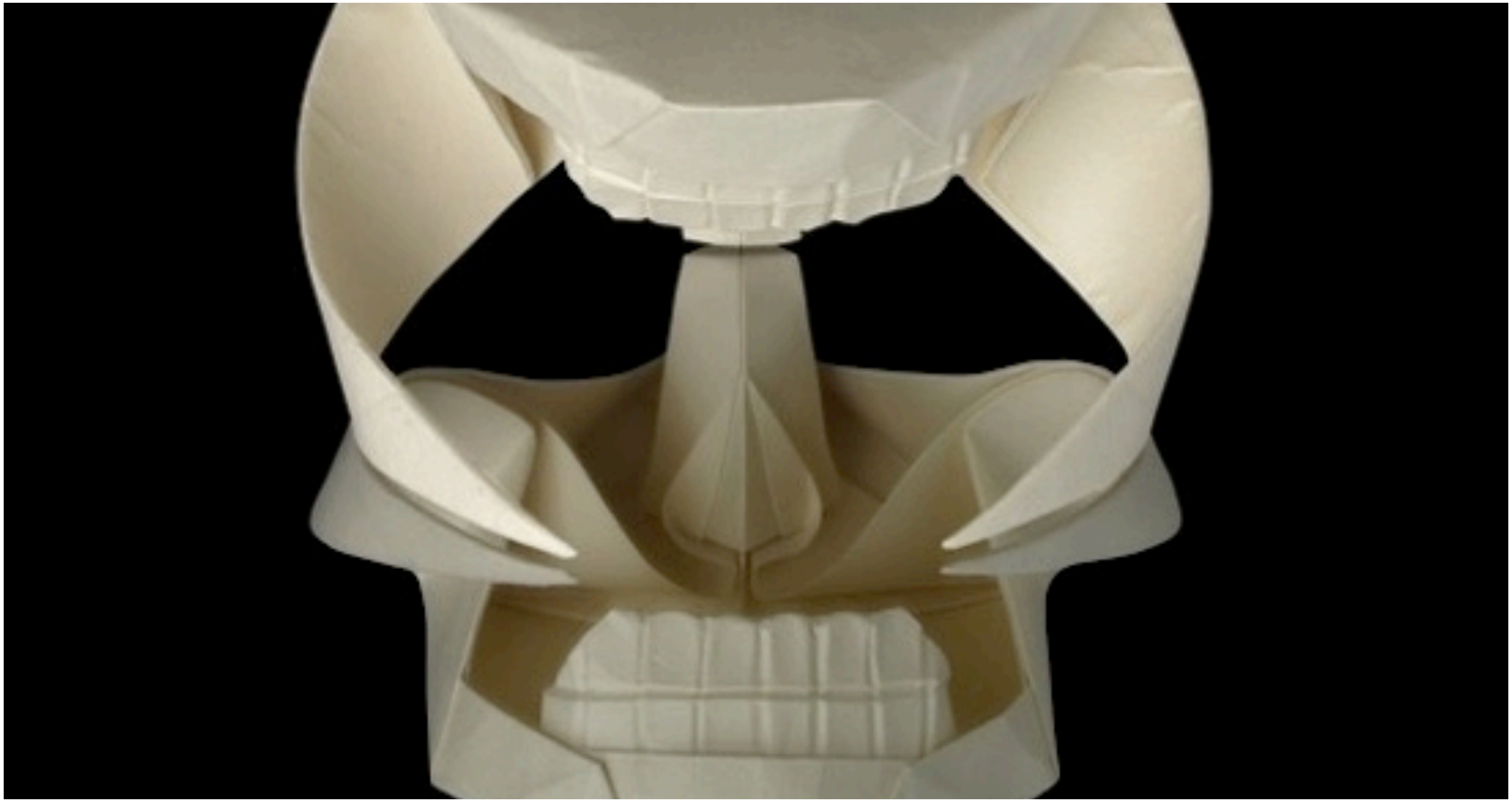
Example of Origami



Herman Van Goubergen's origami Skull

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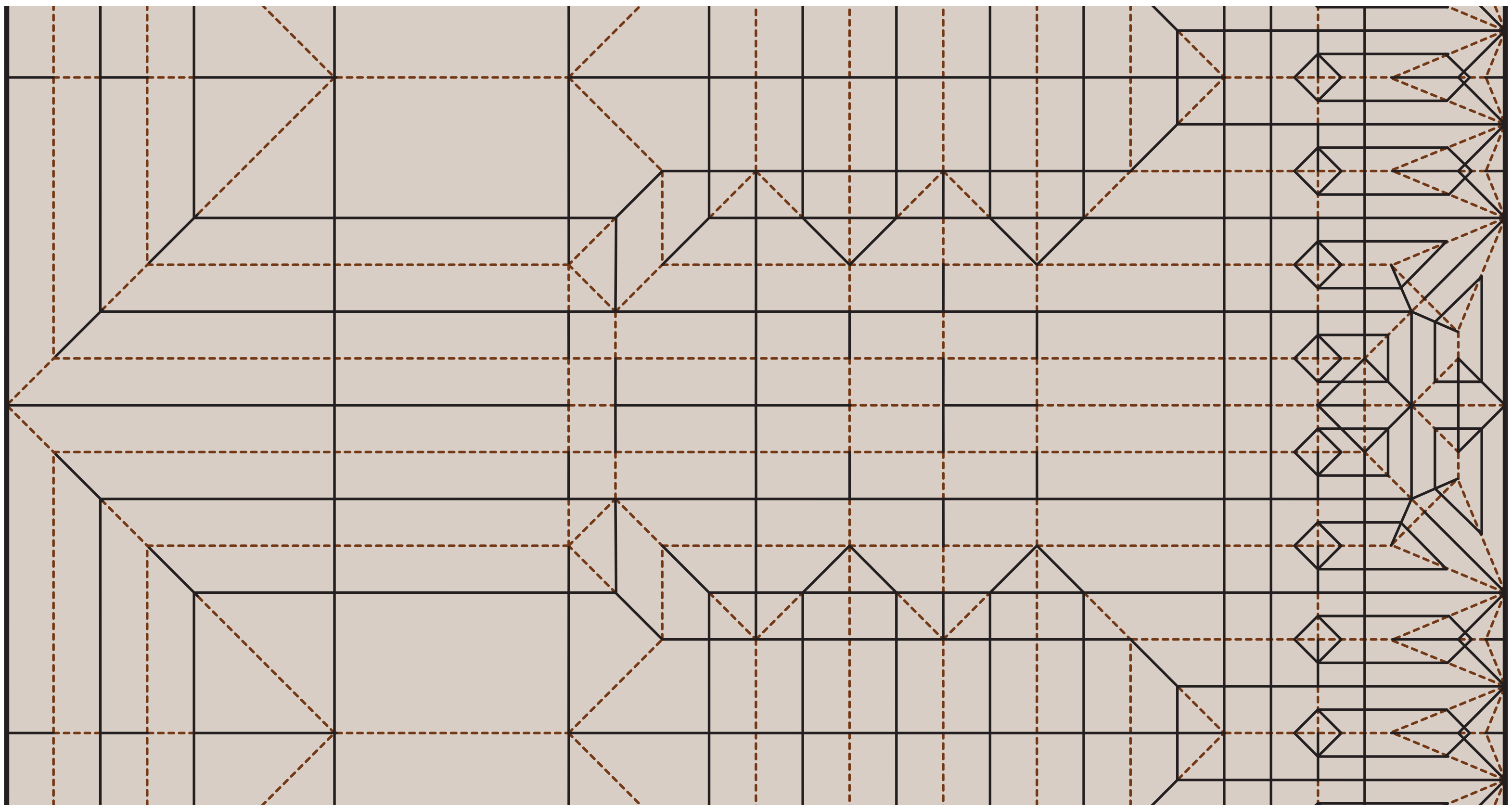
Example of Origami

Robert Lang's Bull Moose http://www.langorigami.com

Example of Origami



Robert Lang's Bull Moose http://www.langorigami.com

Origami Math Revolution

- 1950's Arika Yoshizawa starts writing down origami designs in the modern way.
- 1989-1995 Axioms of Origami discovered by Jacques Justin, Humiaki Huzita, Robert Lang and others.
- 1997-present Origami artists takes advantage of math and computers to find ways to fold new objects.

Paper Experiment

Step 1: Fold your paper in half. Hold it so the fold is vertical.

Step 2: Mark the lower left corner and the bottom point along the fold.

Step 3: Through any marked point P, fold the paper along all the lines passing through P at angle 0^{o},45^{o},90^{o} or 135^{o} to the horizontal line though P. Go to Step 4.

Step 4: Mark every point on the intersection of two folds and repeat Step 3.

Questions

What kinds of patterns do see on your paper?

Is the lower right corner in the set of marked points?

If you folded forevermore, how many marked points would you get on your sheet of paper?

Can you describe the set of all possible marked points?

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Thought Experiment

Step 1: Consider all points in a 2 dimensional plane.

Step 2: Mark the points (0,0) and (1,0).

Step 3: Through any marked point P, draw the line along all passing through P at angle 0^{o}, 45^{o}, 90^{o} or 135^{o} to the horizontal line though P. Go to Step 4.

Step 4: Mark every point on the intersection of two folds and repeat Step 3.

Questions

What kinds of patterns would you see on your plane?

Is the lower right corner in the set of marked points?

If you folded forevermore, how many marked points would you get on your sheet of paper?

Can you describe the set of all possible marked points?

Results

Theorem: The set of points in the plane that can be constructed as marked points in the same way experiment is exactly

$$\left\{\left(\frac{a}{2^j},\frac{b}{2^k}\right): a,b,j,k \ \text{ integers}\right\}$$

Proof

Theorem: The set of points in the plane that can be constructed as marked points in the same way experiment is exactly

$$\left\{\left(\frac{a}{2^j},\frac{b}{2^k}\right): a,b,j,k \text{ integers}\right\}$$

Proof: The point (2,0) is marked. Then given any marked point (x,0), use the same process starting with (1,0)(2,0) to construct (x+1,0). By symmetry, we can also construct (0,2). So every (a,0),(0,b) with a,b integers can be constructed.

Results

Theorem: The set of points in the plane that can be constructed as marked points in the same way experiment is exactly

$$\left\{\left(\frac{a}{2^j},\frac{b}{2^k}\right): a,b,j,k \ \text{ integers}\right\}$$

Proof Continued: Every (a,0) and (0,b) with a,b integers can be constructed. The point (1/2,0) can be constructed. By starting with (0,0) and (1/2,0) we can construct (a/2,0), and by symmetry we can construct (0,b/2). So we can constuct ($a/2^j$,0) and (0,$b/2^k$) for all a,b,j,k integers.

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Proof

Theorem: The set of points in the plane that can be constructed as marked points in the same way experiment is exactly

$$\left\{\left(\frac{a}{2^j},\frac{b}{2^k}\right): a,b,j,k \ \text{ integers}\right\}$$

Proof Continued: We can construct $(a/2^j,0)$ and $(0,b/2^k)$ for all a,b,j,k integers.

If we can construct (x,0) and (0,y) then we can construct (x,y) by starting with the marked points (x,0) and (x+1,0) and applying the folds and intersections to get from (0,0) to (0,y).

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Proof

Theorem: The set of points in the plane that can be constructed as marked points in the same way experiment is exactly

$$\left\{\left(\frac{a}{2^j},\frac{b}{2^k}\right): a,b,j,k \ \text{ integers}\right\}$$

Proof Continued: We can construct $(a/2^j, b/2^k)$ for all a,b,j,k integers.

How do we know we can't get anything else?

Proof

Theorem: The set of points in the plane that can be constructed as marked points in the same way experiment is exactly

$$\left\{ \left(\frac{a}{2^j}, \frac{b}{2^k} \right) : a, b, j, k \ \text{ integers} \right\}$$

Proof Continued: Every new marked point (p,q) after (0,0) and (1,0) is on the intersection of two lines using fixed angles through two previous constructed points (u,v), (x,y). By symmetry, we can assume (p,q) is above both points and by drawing the horizontal line through (u,v) can assume u=x. So q = (y-v)/2 and p = (y-v)/2 + x. So (p,q) is in the set above. QED.

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Results

Main Theorem [Butler-Demaine-Graham-Tachi, 2013]

Fix $n \geq 3$. Starting with the line $y = 0$ and the points $(0, 0)$ and $(1, 0)$ construct new lines and points by using the following two rules. To construct a new line take an existing point and introduce a new line forming an angle of $i\pi/n$ with another line through the point. To construct a new point take the intersection of two lines.

- The set of points that can be constructed for $n = 3, 4, 5, 6, 8, 10, 12, 24$ are given.

n	*Form for constructed points*
3	$a(1,0)+b(\frac{1}{2},\frac{1}{2}\sqrt{3})$
4	$\left(\dfrac{a}{2^k},\dfrac{b}{2^k}\right)$
5	$\left(\dfrac{a+b\sqrt{5}}{2}\right)(1,0)+\left(\dfrac{c+d\sqrt{5}}{2}\right)\left(\dfrac{1}{2},\dfrac{1}{2}\sqrt{5-2\sqrt{5}}\right),$ with $a+b\equiv c+d\equiv 0 \pmod 2$
6	$\left(\dfrac{a}{2^k3^\ell},\dfrac{b}{2^k3^\ell}\sqrt{3}\right)$
8	$\left(\dfrac{a+b\sqrt{2}}{2^k},\dfrac{c+d\sqrt{2}}{2^k}\right)$
10	$\left(\dfrac{a+b\sqrt{5}}{2^k5^\ell},\dfrac{c+d\sqrt{5}}{2^k5^\ell}\sqrt{5-2\sqrt{5}}\right)$
12	$\left(\dfrac{a+b\sqrt{3}}{2^k3^\ell},\dfrac{c+d\sqrt{3}}{2^k3^\ell}\right)$
24	$\left(\dfrac{a+b\sqrt{2}+c\sqrt{3}+d\sqrt{6}}{2^k3^\ell},\dfrac{e+f\sqrt{2}+g\sqrt{3}+h\sqrt{6}}{2^k3^\ell}\right)$

Open Question

Question:[Butler-Demaine-Graham-Tachi, 2013]

Fix $n \geq 3$. Starting with the line $y = 0$ and the points $(0, 0)$ and $(1, 0)$ construct new lines and points by using the following two rules. To construct a new line take an existing point and introduce a new line forming an angle of $i\pi/n$ with another line through the point. To construct a new point take the intersection of two lines.

In this process, how many points can be constructed which need at most m lines to construct?

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Summary

Search for Mathematical Knowledge Everywhere

Experiment with Mathematical Ideas

Create New Theorems

Science Fiction = Scientific Research Problems