Introduction to Affine Grassmannians

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Connections for Women: Algebraic Geometry and Related Fields
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Philosophy

"Combinatorics is the equivalent of nanotechnology in mathematics."

Outline

1. Background and history of Grassmannians
2. Schur functions
3. Background and history of Affine Grassmannians
4. Strong Schur functions and k-Schur functions
5. The Big Picture

New results based on joint work with

- Steve Mitchell (University of Washington) arXiv:0712.2871, 0803.3647
- Sami Assaf (MIT), preprint coming soon!

Enumerative Geometry

Approximately 150 years ago... Grassmann, Schubert, Pieri, Giambelli, Severi, and others began the study of *enumerative geometry*.

Early questions:

- What is the dimension of the intersection between two general lines in $\mathbb{R}^2$?
- How many lines intersect two given lines and a given point in $\mathbb{R}^3$?
- How many lines intersect four given lines in $\mathbb{R}^3$?

Modern questions:

- How many points are in the intersection of 2,3,4,... Schubert varieties in general position?

Schubert Varieties

A *Schubert variety* is a member of a family of projective varieties which is defined as the closure of some orbit under a group action in a homogeneous space G/H.

Typical properties:

- They are all Cohen-Macaulay, some are "mildly" singular.
- They have a nice torus action with isolated fixed points.
- This family of varieties and their fixed points are indexed by combinatorial objects; e.g. partitions, permutations, or Weyl group elements.

Schubert Varieties

"Honey, Where are my Schubert varieties?"

Typical contexts:

- *The Grassmannian Manifold*, $G(n, d) = GL_n/P$.
- The Flag Manifold: Gl_n/B.
- Symplectic and Orthogonal Homogeneous spaces: Sp_{2n}/B, O_n/P
- Homogeneous spaces for semisimple Lie Groups: G/P.
- *Affine Grassmannians*: $\mathcal{L}_G = G(\mathbb{C}[z, z^{-1}])/\widetilde{P}$.

More exotic forms: spherical varieties, Hessenberg varieties, Goresky-MacPherson-Kottwitz spaces.

Why Study Schubert Varieties?

1. It can be useful to see points, lines, planes etc as families with certain properties.
2. Schubert varieties provide interesting examples for test cases and future research in algebraic geometry.
3. Applications in discrete geometry, computer graphics, and computer vision.

The Grassmannian Varieties

Definition. Fix a vector space V over $\mathbb{C}$ (or $\mathbb{R}$, $\mathbb{Z}_2, \ldots$) with basis $B = \{e_1, \ldots, e_n\}$. The *Grassmannian variety*

$$G(k, n) = \{k\text{-dimensional subspaces of } V\}.$$

Question.

How can we impose the structure of a variety or a manifold on this set?

The Grassmannian Varieties

Answer. Relate $G(k,n)$ to the vector space of $k \times n$ matrices.

$$U = \text{span}\langle 6e_1 + 3e_2,\quad 4e_1 + 2e_3,\quad 9e_1 + e_3 + e_4\rangle \in G(3,4)$$

$$M_U = \begin{bmatrix} 6 & 3 & 0 & 0 \\ 4 & 0 & 2 & 0 \\ 9 & 0 & 1 & 1 \end{bmatrix}$$

- $U \in G(k,n) \iff$ rows of M_U are independent vectors in $V \iff$ some $k \times k$ minor of M_U is NOT zero.

Plücker Coordinates

- Define $f_{j_1,j_2,\ldots,j_k}$ to be the polynomial given by the determinant of the matrix
$$\begin{bmatrix} x_{1,j_1} & x_{1,j_2} & \ldots & x_{1,j_k} \\ x_{2,j_1} & x_{2,j_2} & \ldots & x_{2,j_k} \\ \vdots & \vdots & \vdots & \vdots \\ x_{kj_1} & x_{kj_2} & \ldots & x_{kj_k} \end{bmatrix}$$
- $G(k,n)$ is an open set in the Zariski topology on $k \times n$ matrices defined as the complement of the union over all k-subsets of $\{1,2,\ldots,n\}$ of the varieties $V(f_{j_1,j_2,\ldots,j_k})$.
- All the determinants $f_{j_1,j_2,\ldots,j_k}$ are homogeneous polynomials of degree k so $G(k,n)$ can be thought of as a projective variety.

The Grassmannian Varieties

Canonical Form. Every subspace in $G(k,n)$ can be represented by a unique matrix in row echelon form.

Example.

$$U = \text{span}\langle 6e_1 + 3e_2,\quad 4e_1 + 2e_3,\quad 9e_1 + e_3 + e_4\rangle \in G(3,4)$$

$$\approx \begin{bmatrix} 6 & 3 & 0 & 0 \\ 4 & 0 & 2 & 0 \\ 9 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 7 & 0 & 0 & 1 \end{bmatrix}$$

$$\approx \langle 2e_1 + e_2,\quad 2e_1 + e_3,\quad 7e_1 + e_4\rangle$$

Subspaces and Subsets

Example.

$$U = \text{RowSpan} \begin{bmatrix} 5 & 9 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 5 & 8 & 0 & 9 & 7 & 9 & 1 & 0 & 0 & 0 \\ 4 & 6 & 0 & 2 & 6 & 4 & 0 & 3 & 1 & 0 \end{bmatrix} \in G(3,10).$$

$$\text{position}(U) = \{3,7,9\}$$

Definition.

If $U \in G(k,n)$ and M_U is the corresponding matrix in canonical form then the columns of the leading 1's of the rows of M_U determine a subset of size k in $\{1,2,\ldots,n\} := [n]$. There are 0's to the right of each leading 1 and 0's above and below each leading 1. This k-subset determines the *position* of U with respect to the fixed basis.

The Schubert Cell C_{j} in $G(k,n)$

Defn. Let $\mathrm{j} = \{j_1 < j_2 < \cdots < j_k\} \in [n]$. A *Schubert cell* is

$$C_{\mathrm{j}} = \{U \in G(k,n) \mid \mathrm{position}(U) = \{j_1, \ldots, j_k\}\}$$

Fact. $G(k,n) = \bigcup C_{\mathrm{j}}$ over all k-subsets of $[n]$.

Example. In $G(3,10)$,

$$C_{\{3,7,9\}} = \left\{ \begin{bmatrix} * & * & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & 0 & * & * & * & 1 & 0 & 0 & 0 \\ * & * & 0 & * & * & * & 0 & * & 1 & 0 \end{bmatrix} \right\}$$

- Observe, $\dim(C_{\{3,7,9\}}) = 6 + 5 + 2 = 13.$
- In general, $\dim(C_{\mathrm{j}}) = \sum j_i - i.$

Schubert Varieties in $G(k,n)$

Defn. Given $\mathrm{j} = \{j_1 < j_2 < \cdots < j_k\} \in [n]$, the *Schubert variety* is

$$X_\lambda = \text{Closure of } C_\lambda \text{ under Zariski topology.}$$

Question. In $G(3,10)$, which minors vanish on $C_{\{3,7,9\}}$?

$$C_{\{3,7,9\}} = \left\{ \begin{bmatrix} * & * & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & 0 & * & * & * & 1 & 0 & 0 & 0 \\ * & * & 0 & * & * & * & 0 & * & 1 & 0 \end{bmatrix} \right\}$$

Answer. All minors f_{j_1,j_2,j_3} with $\left\{ \begin{array}{c} 4 \leq j_1 \leq 8 \\ \text{or } j_1 = 3 \text{ and } 8 \leq j_2 \leq 9 \\ \text{or } j_1 = 3, j_2 = 7 \text{ and } j_3 = 10 \end{array} \right\}$

In other words, the canonical form for any subspace in $\overline{C_{\mathrm{j}}}$ has 0's to the right of column j_i in each row i.

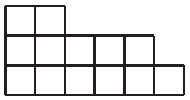
k-Subsets and Partitions

Defn. A *partition* of a number n is a weakly increasing sequence of non-negative integers

$$\lambda = (\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k)$$

such that $n = \sum \lambda_i = |\lambda|$.

Partitions can be visualized by their *Ferrers diagram*

$(2,5,6) \longrightarrow$ 

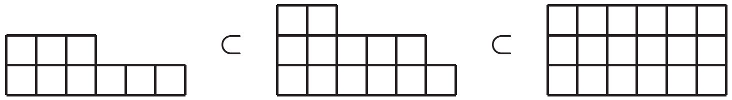
Fact. There is a bijection between k-subsets of $\{1, 2, \ldots, n\}$ and partitions whose Ferrers diagram is contained in the $k \times (n-k)$ rectangle given by

$$\mathrm{shape} : \{j_1 < \ldots < j_k\} \mapsto (j_1 - 1, j_2 - 2, \ldots, j_k - k).$$

A Poset on Partitions

Defn. A *partial order* or a *poset* is a reflexive, anti-symmetric, and transitive relation on a set.

Defn. *Young's Lattice*
If $\lambda = (\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k)$ and $\mu = (\mu_1 \leq \mu_2 \leq \cdots \leq \mu_k)$ then $\lambda \subset \mu$ if the Ferrers diagram for λ fits inside the Ferrers diagram for μ.



Facts.

1. $X_{\mathrm{j}} = \bigcup_{\mathrm{shape(i)} \subset \mathrm{shape(j)}} C_{\mathrm{i}}.$
2. The dimension of X_{j} is $|\mathrm{shape(j)}|$.
3. The Grassmannian $G(k,n) = X_{\{n-k+1,\ldots,n-1,n\}}$ is a Schubert variety!

Enumerative Geometry Revisited

Question. How many lines intersect four given lines in $\mathbb{R}^3$?

Translation. Given a line in $\mathbb{R}^3$, the family of lines intersecting it can be interpreted in $G(2,4)$ as the Schubert variety

$$X_{\{2,4\}} = \overline{\begin{pmatrix} * & 1 & 0 & 0 \\ * & 0 & * & 1 \end{pmatrix}}$$

with respect to a suitably chosen basis determined by the line.

Reformulated Question. How many subspaces $U \in G(2,4)$ are in the intersection of 4 copies of the Schubert variety $X_{\{2,4\}}$ each with respect to a different basis?

Modern Solution. Use Intersection Theory!

Intersection Theory

- Schubert varieties induce canonical basis elements of the cohomology ring for $G(k,n)$ called *Schubert classes*: $[X_{\mathrm{j}}]$.
- Multiplication of Schubert classes corresponds with intersecting Schubert varieties with respect to different bases.

$$[X_{\mathrm{i}}][X_{\mathrm{j}}] = [X_{\mathrm{i}}(B^1) \cap X_{\mathrm{j}}(B^2)]$$

- Schubert classes add and multiply just like *Schur functions*. Schur functions are a fascinating family of symmetric functions indexed by partitions which appear in many areas of math, physics, theoretical computer science.
- Expanding the product of two Schur functions into the basis of Schur functions can be done via linear algebra:

$$S_\lambda S_\mu = \sum c^{\nu}_{\lambda,\mu} S_\nu.$$

- The coefficients $c^{\nu}_{\lambda,\mu}$ are non-negative integers called the Littlewood-Richardson coefficients. In a 0-dimensional intersection, the coefficient of $[X_{\{1,2,\ldots,k\}}]$ is the number of subspaces in $X_{\mathrm{i}}(B^1) \cap X_{\mathrm{j}}(B^2)$.

Schur Functions

Let $X = \{x_1, x_2, \ldots, x_m\}$ be an alphabet.

Let $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0)$ and $\lambda_p = 0$ for $p \geq k$.

Defn. The following are equivalent definitions for the Schur functions $S_\lambda(X)$:

1. $S_\lambda = \frac{\det(x_i^{\lambda_j+n-j})}{\det(x_i^{\lambda_j+n-j})}$ with indices $1 \leq i,j \leq m$.
2. $S_\lambda = \sum x^T$ summed over all *column strict tableaux T of shape λ*.
3. $S_\lambda = \sum Q_{D(T)}(X)$ summed over all *standard tableaux T of shape λ*.

Schur Functions

Defn. $S_\lambda = \sum Q_{D(T)}(X)$ over all *standard tableaux T of shape λ*.

Defn. A standard tableau T of shape λ is a saturated chain in Young's lattice from $\emptyset$ to λ. The descent set of T is the set of indices i such that $i+1$ appears northwest of i.

Example.

$$T = \begin{array}{|c|c|c|c|c|} \hline 7 \\ \cline{1-1} 4 & 5 & 9 \\ \cline{1-3} 1 & 2 & 3 & 6 & 8 \\ \hline \end{array} \qquad D(T) = \{3,6,8\}.$$

Defn. Let $S \subset [p]$. The fundamental quasisymmetric function

$$Q_S(X) = \sum x_{i_1} \cdots x_{i_p}$$

summed over all $1 \leq i_1 \leq \ldots \leq i_p$ such that $i_j < i_{j+1}$ whenever $j \in S$.

Enumerative Solution

Reformulated Question. How many subspaces $U \in G(2,4)$ are in the intersection of 4 copies of the Schubert variety $X_{\{2,4\}}$ each with respect to a different basis?

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Solution.

$$[X_{\{2,4\}}] = S_{(1)} = x_1 + x_2 + \dots$$

By the recipe, compute

$$[X_{\{2,4\}}(B^1) \cap X_{\{2,4\}}(B^2) \cap X_{\{2,4\}}(B^3) \cap X_{\{2,4\}}(B^4)]$$

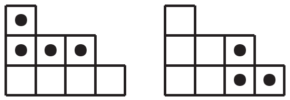
$$= S_{(1)}^4 = 2S_{(2,2)} + S_{(3,1)} + S_{(2,1,1)}.$$

Answer. The coefficient of $S_{2,2} = [X_{1,2}]$ is *2* representing the two lines meeting 4 given lines in general position.

Singularities in Schubert Varieties

Theorem. (Lakshmibai-Weyman) Given a partition λ. The singular locus of the Schubert variety X_λ in $G(k,n)$ is the union of Schubert varieties indexed by the set of all partitions $\mu \subset \lambda$ obtained by removing a hook from λ.

Example. $\mathrm{sing}((X_{(4,3,1)}) = X_{(4)} \cup X_{(2,2,1)}$



Corollary. X_λ is non-singular if and only if λ is a rectangle.

Recap

1. $G(k,n)$ is the Grassmannian variety of k-dim subspaces in $\mathbb{R}^n$.
2. The Schubert varieties in $G(k,n)$ are nice projective varieties indexed by k-subsets of $[n]$ or equivalently by partitions in the $k \times (n-k)$ rectangle.
3. Geometrical information about a Schubert variety can be determined by the combinatorics of partitions.
4. Intersection theory applied to Schubert varieties can be used to solve problems in enumerative geometry.

Grassmannians to Affine Grassmannians

Grassmannians: G/P where $P =$ invertible block upper triangular matrices.

Generalized Grassmannians: G/P where $G =$ simple Lie group,
$P =$ parabolic subgroup.

Affine Grassmannians: $\mathcal{L}_G = \widetilde{G}/\widetilde{P}$ where

- $\widetilde{G}$ is the matrix subgroup G with entries in $\mathbb{C}[z, z^{-1}]$,
- $\widetilde{P} =$ *Iwahori subgroup* = matrix subgroup G with entries in $\mathbb{C}[z]$.

Weyl groups

A *Weyl group* W is a finite reflection group generated by simple reflections $S = \{s_1, \dots, s_n\}$ with relations of the form

$$s_i^2 = 1$$
$$(s_i s_j)^{m_{ij}} = 1 \text{ with } m_{ij} \in \{2, 3, 4, 6\}$$

The relations are encoded in a graph called a *Dynkin diagram* with vertices given by the generators and edges connecting s_i, s_j if $m_{ij} \geq 3$. A single edge implies $m_{ij} = 3$, a double edge implies $m_{ij} = 4$ and a triple edge implies $m_{ij} = 6$.

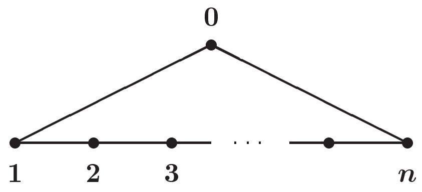
Example. The Symmetric Group S_n is a Weyl group with Dynkin diagram

Affine Weyl groups

For each finite Weyl group W there is an associated *affine Weyl group* $\widetilde{W}$ generated by $S \cup \{s_0\}$ with relations determined by the *affine Dynkin diagram*.

An expression $w = s_{i_1} s_{i_2} \dots s_{i_p}$ is *reduced* if p is minimal over all expressions for w. The *length* of w is $\ell(w) = p$. The *Bruhat order* is a ranked partial order determined by the covering relation $v < w$ if $\ell(v) = \ell(w) - 1$ and $v = s_{i_1} \dots \widehat{s_{i_j}} \cdots s_{i_p}$ for some j.

Example. The Affine Symmetric Group $\widetilde{S}_{n+1}$ has Dynkin diagram



Affine quotients $\widetilde{W}/W$

Defn. Let $\widetilde{W}^S$ be the minimal length coset representatives for $\widetilde{W}/W$. $\widetilde{W}^S$ inherits the length function and Bruhat order from $\widetilde{W}$.

Nice Properties. (Bott, Quillen, Garland-Raghunathan, Mitchell...)

1. The *affine Grassmannian* $\mathcal{L}_G$ is an infinite dimensional variety with the structure of a CW-complex indexed by $\widetilde{W}^S$

$$\mathcal{L}_G = \bigoplus_{w \in \widetilde{W}^S} e_w.$$

where e_w is a cell of complex dimension $\ell(w)$.

2. $\mathcal{L}_G$ is homeomorphic to $\Omega_{\text{alg}} G =$ algebraic based loops of G.

3. *Bott's Formula*: Let $e_1, \dots, e_n$ be the exponents of W. Then

$$\sum H^i(\mathcal{L}_G) t^i = \sum_{v \in \widetilde{W}^S} t^{2\ell(v)} = \prod_{i=1}^{n} \frac{1}{(1 - t^{2e_i})}$$

Example

$\tilde{C}_5$:

$$\widetilde{W}^S(t) = \frac{1}{(1-t)(1-t^3)(1-t^5)(1-t^7)(1-t^9)}$$
$$= 1 + t + t^2 + 2t^3 + 2t^4 + 3t^5 + 4t^6 + 5t^7 + 6t^8 + 8t^9 + \dots$$

Affine Schubert Varieties

Defn. For $\lambda \in \widetilde{W}^S$, define the *affine Schubert variety*

$$X_\lambda = \overline{e_\lambda} = \bigoplus_{\mu \leq \lambda} e_\mu.$$

Defn. Let $P_\lambda =$ Poincaré polynomial for $X_\lambda = \sum_{\mu \leq \lambda} t^{\ell(\mu)}$.

Examples. $\tilde{C}_5$:

$P_{s_2 s_1 s_0} = 1 + t + t^2 + t^3$ *palindromic*

$P_{s_0 s_2 s_1 s_0} = 1 + t + t^2 + 2t^3 + t^4$ *not palindromic*

Affine Schubert Varieties

Questions. Which X_λ are smooth? Which X_λ are palindromic?

Implications.

X_λ smooth $\implies$ X_λ satisfies Poincaré duality integrally
$\implies$ X_λ satisfies Poincaré duality rationally
$\iff$ X_λ is palindromic by Carrell-Peterson theorem.

Obvious Candidates. X_λ is a *closed parabolic orbit*, if there exists a parabolic subgroup $P_I \subset \tilde{G}$ such that

$$X_\lambda = P_I \cdot e_{\mathrm{id}}.$$

These affine Schubert varieties are generalized Grassmannians G/P, hence they are smooth.

Smooth Affine Schubert Varieties

Theorem. (Billey-Mitchell) The following are equivalent:

1. X_λ is smooth.
2. X_λ satisfies Poincaré duality integrally.
3. X_λ closed parabolic orbit for some $I \subsetneq \tilde{S}$.
4. For all $i \in I$, $s_i\lambda < \lambda$.

Palindromic Affine Schubert Varieties

Theorem. (Billey-Mitchell) X_λ is palindromic iff one of the following hold:

1. For all $i \in I$, $s_i\lambda < \lambda$. (closed parabolic orbit)
2. X_λ is a *chain*, i.e. $P_\lambda = \sum_{i=0}^{\ell(\lambda)} t^i$
3. Type $= B_3$ and $\lambda = s_3s_2s_0 \cdot s_3s_2s_1 \cdot s_3s_2s_0$.
4. Type $= A_n$ and X_λ is a *spiral variety*. or $X'_{k,n}$.

Defn. X_λ is a *spiral variety* in type A_n if λ has a single reduced expression and length kn for some k. (previously studied by Mitchell, Cohen-Lupercio-Segal)

Two Corollaries

Corollary. In every Weyl group type, there are only a finite number of non-trivial smooth affine Schubert varieties corresponding with connected subgraphs of the Dynkin diagram containing s_0.

$$\begin{array}{llll} A_n: & \binom{n+1}{2} & E_{6,7,8}: & 10 \\ B_n: & 2n-2 & F_4: & 5 \\ C_n: & n & G_2: & 3 \\ D_n: & 2n & & \end{array}$$

Corollary. Smooth is equivalent to palindromic for all affine Schubert varieties of *non-cyclic simply laced* affine types only, i.e. $\tilde{D}, \tilde{E}$.

Corollary. Outside of type A, there are only finitely many palindromic X_λ.

(Co)Homology for Affine Grassmannians

Theorem.(Bott) The homology $H_*(\mathcal{L}_G)$ and cohomology $H^*(\mathcal{L}_G)$ form dual Hopf algebras isomorphic to a subring and a quotient ring of the symmetric functions.

Question. *What is analog of Schur functions for $H_*(\mathcal{L}_G)$ and $H^*(\mathcal{L}_G)$?*

(Co)Homology for Affine Grassmannians

Theorem.(Bott) The homology $H_*(\mathcal{L}_G)$ and cohomology $H^*(\mathcal{L}_G)$ form dual Hopf algebras isomorphic to a subring $\Lambda_{(n)}$ and a quotient ring $\Lambda^{(n)}$ of the symmetric functions.

Question. *What is analog of Schur functions for $H_*(\mathcal{L}_G)$ and $H^*(\mathcal{L}_G)$?*

- Affine Schubert varieties induce classes in the homology/cohomology for $\mathcal{L}_G$. These classes form dual Hopf algebra bases.
- What family of symmetric functions form a basis for $\Lambda_{(n)}$ and $\Lambda^{(n)}$ and multiply like these classes?

(Co)Homology for Affine Grassmannians

Theorem. (Lam-Lapointe-Morse-Shimozono) For $G = SL_n$, the homology classes and cohomology classes for affine Schubert varieties are the strong Schur functions and weak Schur functions respectively.

Nice Properties.

1. Both families are symmetric functions.
2. The strong Schur functions are defined via saturated chains in Bruhat order on $\widetilde{S}_n/S_n$. They are evaluations of the k-Schur functions at $t = 1$ using the definition of Lapointe-Lascoux-Morse.
3. The weak Schur functions are defined in terms of the weak order on $\widetilde{S}_n/S_n$. They are related to Stanley symmetric functions.
4. (Lam-Shimozono-Schilling) Similar results hold for type C.

Strong Schur and k-Schur Functions

Defn. (LLMS) Given $\lambda \in \widetilde{S}_n/S_n$ with $\ell(\lambda) = q$ the *strong Schur function* index by λ is

$$Strong_\lambda^{(n)}(X) = \sum Q_{D(m_1,m_2,\ldots,m_q)}$$

first summed over all labeled saturated chains $(C, m_1, m_2, \ldots, m_q)$ from id to λ in Bruhat order

LLMS also defined a spin statistic on the labeled chain.

Defn. (LLMS) Let $k = n-1$, $\lambda \in \widetilde{S}_n/S_n$, $\ell(\lambda) = q$ the *k-Schur function* indexed by λ is

$$S_\lambda^{(k)}(X) = \sum t^{\mathrm{spin}(C,m_1,\ldots,m_q)} Q_{D(m_1,m_2,\ldots,m_q)}$$

over the same sum.

More details on Sami Assaf's poster this afternoon!

Schur Positivity

Theorem. (Assaf-Billey) Using this definition the k-Schur function $S_\lambda^{(k)}(X)$ expands as a positive sum of Schur functions.

Benefits.

- Significantly reduces number of terms in the expansion so easier to store and manipulate.
- Simplifies computations of products in the homology ring.
- Each k-Schur can be associated to an S_n-module.

Open. Is there nice analog of the Littlewood-Richardson rule for k-Schurs?

Proof Techniques

Theorem. (Assaf) Say $F(X) = \sum_{S \in \mathcal{A}} Q_S$.
If one can impose a *D graph structure* on $\mathcal{A}$ then $F(X)$ is Schur positive.

Theorem. (Misra-Miwa,Kerber-James,Lapointe-Morse)
There is a rank preserving bijection from $\widetilde{S}_n/S_n$ to n-core partitions (no hook lengths are multiples of n)

$$\mathcal{C} : \widetilde{S}_n/S_n \longrightarrow n\text{-core partitions}$$

Theorem. (Lascoux) Let $v, w \in \widetilde{S}_n/S_n$.
Then $v \leq w$ in Bruhat order if and only if $\mathcal{C}(v) \subset \mathcal{C}(w)$ in Young's lattice.

Big Picture

Geometry	Combinatorics	Rep Theory
Grassmannians	Schur functions	S_n, GL_n irreducible reps
Affine Grassmannians	k-Schur	???? (Li-Chung Chen + Haiman)
Hilbert Schemes of points in plane	Macdonald polynomials	Garsia-Haiman module (n!-theorem)

Big Picture

Geometry	Combinatorics	Rep Theory
Grassmannians	Schur functions $\uparrow$	S_n, GL_n irreducible reps
Affine Grassmannians	k-Schur $\uparrow$	???? (Li-Chung Chen + Haiman)
Hilbert Schemes of points in plane	Macdonald polynomials	Garsia-Haiman module (n!-theorem)

Theorem.(A-B) k-Schur functions are Schur positive.

Conjecture.(Lapoint-Morse) Macdonald polynomials are k-Schur positive for the right k.

Open. Find a direct geometric connection from Hilbert Schemes to Affine Grassmannians.